MATH2103: Lecture Note on Numerical Solution of Partial Differential Equations

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Contents

Chapter 1

The Construction of a Finite Element Space

To approximate the solution of the variational problem,

$$a(u, v) = F(v) \quad \forall v \in V,$$

developed in Chapter 0, we need to construct finite-dimensional subspaces $S \subset V$ in a systematic, practical way.

Let us examine the space S defined in Sect. 0.4. To understand fully the functions in the space S, we need to answer the following questions:

1. **What does a function look like in a given subinterval?**
2. **How do we determine the function in a given subinterval?**
3. **How do the restrictions of a function on two neighboring intervals match at the common boundary?**

In this chapter, we will define piecewise function spaces that are similar to S, but which are defined on more general regions. We will develop concepts that will help us answer these questions.

1.1 3.1 The Finite Element

We follow Ciarlet's definition of a finite element (Ciarlet 1978).

Definition 1.1.1 ((3.1.1)) *Let*

(i) $K \subseteq \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary (the ***element domain****),*

(ii) $\mathcal{P}$ be a finite-dimensional space of functions on K (the ***space of shape functions****) and*

(iii) $\mathcal{N} = \{N_1, N_2, \ldots, N_k\}$ be a basis for $\mathcal{P}'$ (the ***set of nodal variables****).*

Then $(K, \mathcal{P}, \mathcal{N})$ is called a ***finite element****.*

It is implicitly assumed that the nodal variables, N_i, lie in the dual space of some larger function space, e.g., a Sobolev space.

Definition 1.1.2 ((3.1.2)) *Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{N}$* ***(i.e.,** $N_i(\phi_j) = \delta_{ij}$**)** *is called the* ***nodal basis*** *of $\mathcal{P}$.*

Example 1.1.3 ((3.1.3) (the 1-dimensional Lagrange element)) *Let $K = [0, 1]$, $\mathcal{P}$ = the set of linear polynomials and $\mathcal{N} = \{N_1, N_2\}$, where $N_1(v) = v(0)$ and $N_2(v) = v(1)$ $\forall v \in \mathcal{P}$. Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element and the nodal basis consists of $\phi_1(x) = 1 - x$ and $\phi_2(x) = x$.*

Example 1.1.4 *In general, we can let $K = [a, b]$ and $\mathcal{P}_k$ = the set of all polynomials of degree less than or equal to k. Let $\mathcal{N}_k = \{N_0, N_1, N_2, \ldots, N_k\}$, where $N_i(v) = v(a + (b - a)i/k)$ $\forall v \in \mathcal{P}_k$ and $i = 0, 1, \ldots, k$. Then $(K, \mathcal{P}_k, \mathcal{N}_k)$ is a finite element. The verification of this uses Lemma 3.1.4.*

Usually, **condition (iii) of Definition 3.1.1 is the only one that requires much work**, and the following simplifies its verification.

Lemma 1.1.5 ((3.1.4)) *Let P be a d-dimensional vector space and let $\{N_1, N_2, \ldots, N_d\}$ be a subset of the dual space $\mathcal{P}'$. Then the following two statements are equivalent.*

(a) $\{N_1, N_2, \ldots, N_d\}$ is a basis for $\mathcal{P}'$.

(b) Given $v \in \mathcal{P}$ with $N_i(v) = 0$ for $i = 1, 2, \ldots, d$, then $v \equiv 0$.

Remark 1.1.6 ((3.1.7)) *Condition (iii) of Definition 3.1.1 is the same as (a) in Lemma 3.1.4, which can be verified by checking (b) in Lemma 3.1.4. For instance, in Example 3.1.3, $v \in \mathcal{P}_1$ means $v = a + bx$; $N_1(v) = N_2(v) = 0$ means $a = 0$ and $a + b = 0$. Hence, $a = b = 0$, i.e., $v \equiv 0$. More generally, if $v \in \mathcal{P}_k$ and $0 = N_i(v) = v(a + (b - a)i/k)$ for all $i = 0, 1, \ldots, k$ then v vanishes identically by the fundamental theorem of algebra. Thus, $(K, \mathcal{P}_k, \mathcal{N}_k)$ is a finite element.*

$N_1(v) = v(0) \Rightarrow N_1$为$v$在0处取值；$N_2(v) = v(1) \Rightarrow N_2$为$v$在1处取值；

1.2 3.2 Triangular Finite Elements

Let K be any triangle. Let $\mathcal{P}_k$ denote the set of all polynomials in two variables of degree $\leq k$. The following table gives the dimension of $\mathcal{P}_k$.

Table 1.1: Dimension of $\mathcal{P}_k$ in two dimensions

k	$\dim \mathcal{P}_k$
1	3
2	6
3	10
$\vdots$	$\vdots$
k	$\frac{1}{2}(k+1)(k+2)$

The Lagrange Element

Example 1.2.1 ((3.2.1) $(k = 1)$) *Let $\mathcal{P} = \mathcal{P}_1$. Let $\mathcal{N} = \{N_1, N_2, N_3\}$ $(\dim \mathcal{P}_1 = 3)$ where $N_i(v) = v(z_i)$ and z_1, z_2, z_3 are the vertices of K. This element is depicted in Fig. 3.1.*

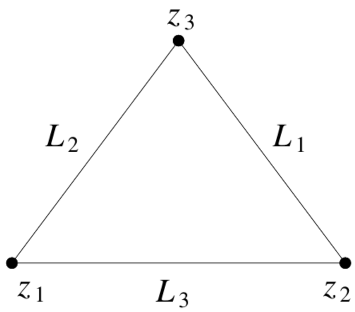


Fig. 3.1. linear Lagrange triangle

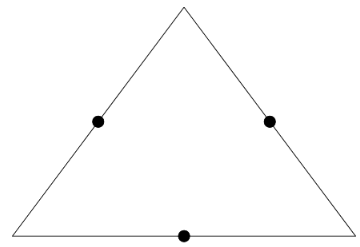


Fig. 3.2. Crouzeix-Raviart nonconforming linear triangle

Figure 1.1: **Fig. 3.1.** linear Lagrange triangle. **Fig. 3.2.** Crouzeix-Raviart nonconforming linear triangle.

Note that "•" indicates the nodal variable evaluation at the point where the dot is located.

We verify 3.1.1(iii) using 3.1.4(b), i.e., we prove that $\mathcal{N}_1$ determines $\mathcal{P}_1$. Let L_1, L_2 and L_3 be non-trivial linear functions that define the lines on which lie the edges of the triangle. Suppose that a polynomial $P \in \mathcal{P}$ vanishes at z_1, z_2 and z_3. Since $P|_{L_1}$ is a linear function of one variable that vanishes at two points, $P = 0$ on L_1. By Lemma 3.1.10 we can write $P = c\,L_1$, where c is a constant. But

$$0 = P(z_1) = c\,L_1(z_1) \implies c = 0$$

(because $L_1(z_1) \neq 0$). Thus, $P \equiv 0$ and hence $\mathcal{N}_1$ determines $\mathcal{P}_1$.

Remark 1.2.2 ((3.2.2)) *The above choice for $\mathcal{N}$ is not unique. For example, we could have defined the* ***non-conforming***

$$N_i(v) = v(\textit{midpoint of the } i^{th} \textit{ edge}),$$

as shown in Fig. 3.2. By connecting the midpoints, we construct a triangle on which $P \in \mathcal{P}_1$ vanishes at the vertices. An argument similar to the one in Example 3.2.1 shows that $P \equiv 0$ and hence, $\mathcal{N}_1$ determines $\mathcal{P}_1$.

Example 1.2.3 ((3.2.3) $(k = 2)$) *Let $\mathcal{P} = \mathcal{P}_2$. Let $\mathcal{N}_2 = \{N_1, N_2, \ldots, N_6\}$ $(\dim \mathcal{P}_2 = 6)$ where*

$$N_i(v) = \begin{cases} v(i^{th} \textit{ vertex}), & i = 1, 2, 3; \\ \\ v(\textit{midpoint of the } (i-3) \textit{ edge}), & \\ \textit{(or any other point on the } i-3 \textit{ edge)} & i = 4, 5, 6. \end{cases}$$

This element is depicted in Fig. 3.3.

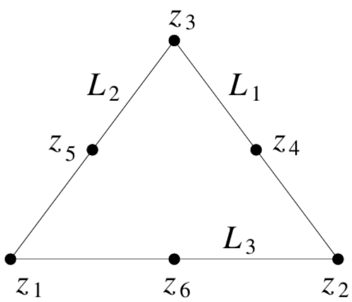


Fig. 3.3. quadratic Lagrange triangle

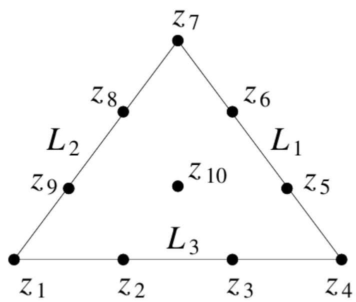


Fig. 3.4. cubic Lagrange triangle

Figure 1.2: **Fig. 3.3.** quadratic Lagrange triangle. **Fig. 3.4.** cubic Lagrange triangle.

We need to check that $\mathcal{N}_2$ determines $\mathcal{P}_2$. As before, let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle. Suppose that the polynomial $P \in P_2$ vanishes at $z_1, z_2, \ldots, z_6$. Since $P|_{L_1}$ is a quadratic function of one variable that vanishes at three points, $P = 0$ on L_1. By Lemma 3.1.10 we can write $P = L_1 Q_1$ where $\deg Q_1 = (\deg P) - 1 = 2 - 1 = 1$. But P also vanishes on L_2. Therefore, $L_1 Q_1|_{L_2} = 0$. Hence, on L_2, either $L_1 = 0$ or $Q_1 = 0$. But L_1 can equal zero only at one point of L_2 since we have a non-degenerate triangle. Therefore, $Q_1 = 0$ on L_2, except possibly at one point. By continuity, we have $Q_1 \equiv 0$ on L_2.

By Lemma 3.1.10, we can write $Q_1 = L_2 Q_2$, where $\deg Q_2 = (\deg L_2) - 1 = 1 - 1 = 0$. Hence, Q_2 is a constant (say c), and we can write $P = cL_1 L_2$. But $P(z_6) = 0$ and z_6 does not lie on either L_1 or L_2. Therefore,

$$0 = P(z_6) = cL_1(z_6)L_2(z_6) \implies c = 0,$$

since $L_1(z_6) \neq 0$ and $L_2(z_6) \neq 0$. Thus, $P \equiv 0$. $\square$

Example 1.2.4 ((3.2.4) ($k = 3$)) *Let $\mathcal{P} = P_3$. Let $\mathcal{N}_3 = \{N_i : i = 1, 2, \ldots, 10 \ (= \dim \mathcal{P}_3)\}$ where*

$$N_i(v) = v(z_i), \quad i = 1, 2, \ldots, 9 \, (z_i \textit{ distinct points on edges as in Fig. 3.4})$$

and

$$N_{10}(v) = v(\textit{any interior point}).$$

We must show that $\mathcal{N}_3$ determines $\mathcal{P}_3$.

In general for $k \geq 1$, we let $\mathcal{P} = \mathcal{P}_k$. For $\mathcal{N}_k = \{N_i : i = 1, 2, \ldots, \frac{1}{2}(k+1)(k+2)\}$, we choose evaluation points at

$$\begin{aligned} &3 \text{ vertex nodes,} \\ &3(k-1) \text{ distinct edge nodes and} \\ &\frac{1}{2}(k-2)(k-1) \text{ interior points.} \end{aligned}$$

(The interior points are chosen, by induction, to determine $\mathcal{P}_{k-3}$). Note that these choices suffice since

$$\begin{aligned}3+3(k-1)+\frac{1}{2}(k-2)(k-1) &= 3k+\frac{1}{2}(k^2-3k+2)\\ &= \frac{1}{2}(k^2+3k+2)\\ &= \frac{1}{2}(k+1)(k+2)\\ &= \dim \mathcal{P}_k.\end{aligned}$$

The evaluation points for $k=4$ and $k=5$ are depicted in Fig. 3.5. **Fig. 3.5.** Lagrange triangles for $k=4$ (left) and $k=5$ (right).

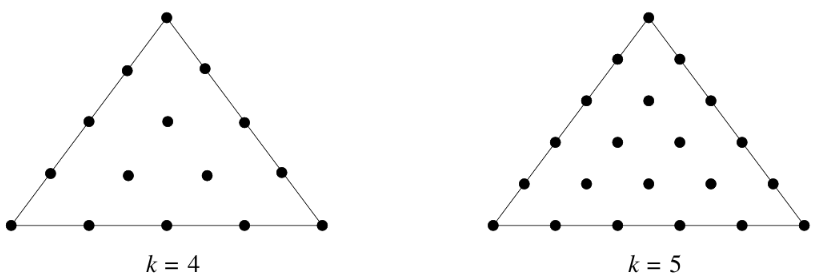


Fig. 3.5. quartic and quintic Lagrange triangles

Figure 1.3: **Fig. 3.5.** quartic and quintic Lagrange triangles.

To show that $\mathcal{N}_k$ determines $\mathcal{P}_k$, we suppose that $P \in \mathcal{P}_k$ vanishes at all the nodes. Let L_1, L_2 and L_3 be non-trivial linear functions that define the edges of the triangle. As before, we conclude from the vanishing of P at the edge and vertex nodes that $P = Q\,L_1\,L_2\,L_3$ where $\deg(Q) \le k-3$; Q must vanish at all the interior points, since none of the L_i can be zero there. These points were chosen precisely to determine that $Q \equiv 0$.

The Hermite Element

Example 1.2.5 ((3.2.6) ($k=3$ Cubic Hermite)) *Let $\mathcal{P} = \mathcal{P}_3$. Let "●" denote evaluation at the point and "○" denote evaluation of the gradient at the center of the circle. Note that the latter corresponds to two distinct nodal variables, but the particular representation of the gradient is not unique. We claim that $\mathcal{N} = \{N_1, N_2, \ldots, N_{10}\}$, as depicted in Fig. 3.6, determines $\mathcal{P}_3$ ($\dim \mathcal{P}_3 = 10$).*

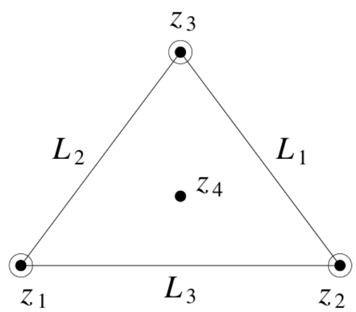


Fig. 3.6. cubic Hermite triangle

Figure 1.4: **Fig. 3.6.** cubic Hermite triangle.

Remark 1.2.6 ((3.2.7)) *Using directional derivatives, there are various distinct ways to define a finite element using $\mathcal{P}_3$, two of which are shown in Fig. 3.7. Note that arrows represent* ***directional derivatives*** *along the indicated directions at the points. The "global" element to the left has the advantage of ease of computation of directional derivatives in the x or y directions throughout the larger region divided up into triangles. The "local" element to the right holds the advantage in that the nodal parameters of each triangle are invariant with respect to the triangle.*

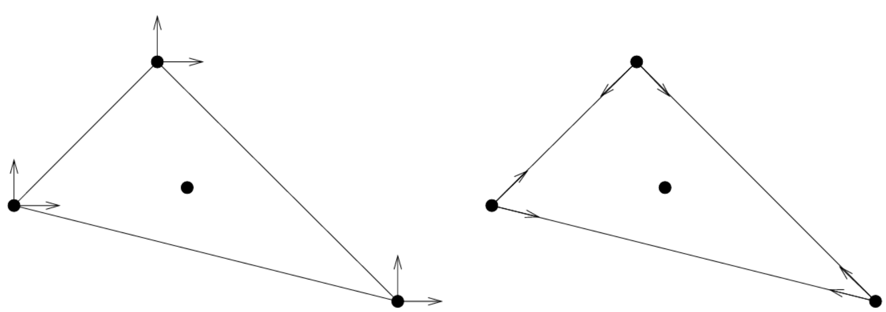


Fig. 3.7. Two different sets of nodal values for cubic Hermite elements.

Figure 1.5: **Fig. 3.7.** two cubic Hermite variants.

In the general Hermite case, we have

$$\begin{cases} 3 \text{ vertex nodes} \\ 6 \text{ directional derivatives (2 for each gradient, evaluated at each of the 3 vertices)} \\ 3(k-3) \text{ edge nodes} \\ \frac{1}{2}(k-2)(k-1) \text{ interior nodes (as in the Lagrange case).} \end{cases} \tag{3.2.9}$$

Note that these sum to $\frac{1}{2}(k+1)(k+2) = \dim \mathcal{P}_k$ as in the Lagrange case.

Example 1.2.7 ((3.2.8) ($k = 4$)) *We have ($\dim \mathcal{P}_4 = 15$). Then $\mathcal{N} = \{N_1, N_2, \ldots, N_{15}\}$, as depicted in Fig. 3.8, determines $\mathcal{P}_4$.*

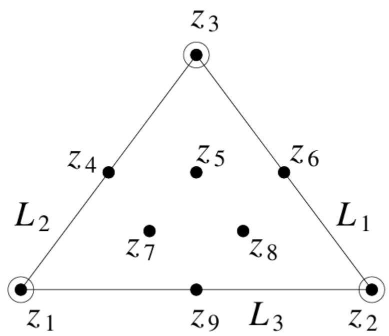


Fig. 3.8. quartic Hermite triangle

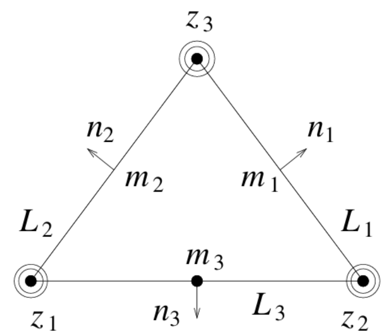


Fig. 3.9. quintic Argyris triangle

Figure 1.6: **Fig. 3.8.** quartic Hermite triangle. **Fig. 3.9.** quintic Argyris triangle.

The Argyris Element

Example 1.2.8 ((3.2.10) ($k = 5$)) *Let $\mathcal{P} = \mathcal{P}_5$. Consider the 21 (= $\dim \mathcal{P}_5$) degrees of freedom shown in Fig. 3.9. As before, let $\bullet$ denote evaluation at the point and the inner circle denote evaluation of the gradient at the center. The outer circle denotes evaluation of the three second derivatives at the center. The arrows represent the evaluation of the normal derivatives at the three midpoints. We claim that*

$$\mathcal{N} = \{N_1, N_2, \ldots, N_{21}\}$$

determines $\mathcal{P}_5$.

Suppose that for some $P \in \mathcal{P}_5$,

$$N_i(P) = 0$$

for $i = 1, 2, \ldots, 21$. Let L_i be as before in the Lagrange and Hermite cases. The restriction of P to L_1 is a fifth order polynomial in one variable with triple roots at z_2 and z_3. Hence, P vanishes identically on L_1. Similarly, P vanishes on L_2 and L_3. Therefore, $P = QL_1L_2L_3$, where $\deg Q = 2$. Observe that $(\partial_{L_1}\partial_{L_2}P)(z_3) = 0$, where ∂_{L_1} and ∂_{L_2} are the directional derivatives along L_1 and L_2 respectively. Therefore,

$$0 = (\partial_{L_1}\partial_{L_2}P)(z_3) = Q(z_3)L_3(z_3)\partial_{L_2}L_1\partial_{L_1}L_2,$$

since

$$\partial_{L_i}L_i \equiv 0 \quad \& \quad L_i(z_3) = 0, \quad i = 1, 2.$$

This implies

$$Q(z_3) = 0$$

because

$$L_3(z_3) \neq 0, \quad \partial_{L_2}L_1 \neq 0$$

and

$$\partial_{L_1}L_2 \neq 0.$$

Similarly,

$$Q(z_1) = 0$$

and

$$Q(z_2) = 0.$$

Also, since

$$L_1(m_1) = 0, \quad \frac{\partial}{\partial n_1} P(m_1) = \left(Q \frac{\partial L_1}{\partial n_1} L_2 L_3 \right)(m_1).$$

Therefore,

$$0 = \frac{\partial}{\partial n_1} P(m_1) \implies Q(m_1) = 0$$

because

$$\frac{\partial L_1}{\partial n_1} \neq 0, \quad L_2(m_1) \neq 0$$

and

$$L_3(m_1) \neq 0.$$

Similarly,

$$Q(m_2) = 0$$

and

$$Q(m_3) = 0.$$

So

$$Q \equiv 0$$

by Example 3.2.3.

1.3 3.3 The Interpolant

Now that we have examined a number of finite elements, we wish to piece them together to create subspaces of Sobolev spaces. We begin by defining the (local) interpolant.

Definition 1.3.1 ((3.3.1)) *Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i : 1 \le i \le k\} \subseteq \mathcal{P}$ be the basis dual to $\mathcal{N}$. If v is a function for which all $N_i \in \mathcal{N}$, $i = 1, \ldots, k$, are defined, then we define the local interpolant by*

$$\mathcal{I}_K v := \sum_{i=1}^{k} N_i(v)\, \phi_i. \tag{3.3.2}$$

Example 1.3.2 ((3.3.3)) *Let K be the triangle depicted in Fig. 3.11, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{N_1, N_2, N_3\}$ as in Example 3.2.1, and $f = e^{xy}$. We want to find $\mathcal{I}_K f$.*

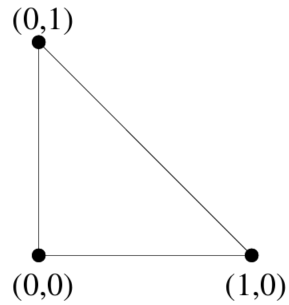


Fig. 3.11. coordinates for linear interpolant

Figure 1.7: **Fig. 3.11.** coordinates for linear interpolant.

By definition, $\mathcal{I}_K f = N_1(f)\phi_1 + N_2(f)\phi_2 + N_3(f)\phi_3$. *We must therefore determine* ϕ_1, ϕ_2 *and* ϕ_3. *The line* L_1 *is given by* $y = 1 - x$. *We can write* $\phi_1 = cL_1 = c(1 - x - y)$. *But* $N_1(\phi_1) = 1$ *implies that* $c = \phi_1(z_1) = 1$, *hence* $\phi_1 = 1 - x - y$. *Similarly,* $\phi_2 = L_2(x, y)/L_2(z_2) = x$ *and* $\phi_3 = L_3(x, y)/L_3(z_3) = y$. *Therefore,*

$$\begin{aligned}\mathcal{I}_K f &= N_1(f)(1 - x - y) + N_2(f)x + N_3(f)y \\ &= 1 - x - y + x + y \qquad (\textit{since } f = e^{xy}) \\ &= 1.\end{aligned}$$

Properties of the interpolant follow.

Proposition 1.3.3 ((3.3.4)) $\mathcal{I}_K$ *is linear.*
Proof. *See exercise 3.x.2.* ■

Proposition 1.3.4 ((3.3.5)) $N_i(\mathcal{I}_K(f)) = N_i(f) \quad \forall 1 \le i \le k$.
Proof. *We have*

$$\begin{aligned}N_i(\mathcal{I}_K(f)) &= N_i\left(\sum_{j=1}^{k} N_j(f)\phi_j\right) \quad (\textit{definition of } \mathcal{I}_K(f)) \\ &= \sum_{j=1}^{k} N_j(f) N_i(\phi_j) \quad (\textit{linearity of } N_i) \\ &= N_i(f) \quad (\{\phi_j\} \textit{ dual to } \{N_j\}).\end{aligned}$$

■

Remark 1.3.5 ((3.3.6)) *Proposition 3.3.5 has the interpretation that* $\mathcal{I}_K(f)$ *is the unique shape function that has the same nodal values as* f.

Proposition 1.3.6 ((3.3.7)) $\mathcal{I}_K(f) = f$ *for* $f \in \mathcal{P}$. *In particular,* $\mathcal{I}_K$ *is idempotent, i.e.,* $\mathcal{I}_K^2 = \mathcal{I}_K$.
Proof. *From (3.3.5),*

$$N_i(f - \mathcal{I}_K(f)) = 0 \quad \forall i$$

which implies the first assertion. The second is a consequence of the first:

$$\mathcal{I}_K^2 f = \mathcal{I}_K(\mathcal{I}_K f) = \mathcal{I}_K f,$$

since $\mathcal{I}_K f \in P$. ■

We now piece together the elements.

Definition 1.3.7 ((3.3.8)) *A* ***subdivision* 剖分** *of a domain* Ω *is a finite collection of element domains* $\{K_i\}$ *such that*

1. $\operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset$ *if* $i \neq j$ *and*
2. $\bigcup K_i = \overline{\Omega}$.

Definition 1.3.8 ((3.3.9)) *Suppose* Ω *is a domain with a subdivision* $\mathcal{T}$*. Assume each element domain,* K*, in the subdivision is equipped with some type of shape functions,* $\mathcal{P}$*, and nodal variables,* $\mathcal{N}$*, such that* $(K, \mathcal{P}, \mathcal{N})$ *forms a finite element. Let* m *be the order of the highest partial derivatives involved in the nodal variables. For* $f \in C^m(\overline{\Omega})$*, the global interpolant is defined by*

$$\mathcal{I}_T f|_{K_i} = \mathcal{I}_{K_i} f \tag{3.3.10}$$

for all $K_i \in \mathcal{T}$.

Without further assumptions on a subdivision, no continuity properties can be asserted for the global interpolant. We now describe conditions that yield such continuity. Only the two-dimensional case using triangular elements is considered in detail here; analogous definitions and results can be formulated for higher dimensions and other subdivisions.

Definition 1.3.9 ((3.3.11)) *A triangulation of a polygonal domain* Ω *is a subdivision consisting of triangles having the property that*

3. *no vertex of any triangle lies in the interior of an edge of another triangle.*

Example 1.3.10 ((3.3.12)) *The figure on the left of Fig. 3.12 shows a triangulation of the given domain. The figure on the right is not a triangulation.*

Fig. 3.12. Two subdivisions: the one on the left is a triangulation and the one on the right is not.

Figure 1.8: **Fig. 3.12.** Two subdivisions: the one on the left is a triangulation and the right one is NOT.

Example 1.3.11 ((3.3.13)) *Let Ω be the square depicted in Fig. 3.13. The triangulation $\mathcal{T}$ consists of the two triangles T_1 and T_2, as indicated. The finite element on each triangle is the Lagrange element in Example 3.2.1. The dual basis on T_1 is $\{1-x-y, x, y\}$ (calculated in Example 3.3.3) and the dual basis on T_2 is (cf. exercise 3.x.3) $\{1-x, 1-y, x+y-1\}$. Let $f = \sin(\pi(x+y)/2)$. Then*

$$\mathcal{I}_T f = \begin{cases} x+y & \text{on } T_1, \\ 2-x-y & \text{on } T_2. \end{cases}$$

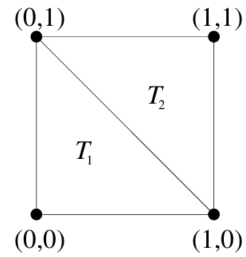


Fig. 3.13. simple triangulation consisting of two triangles

Figure 1.9: **Fig. 3.13.** simple triangulation on square domain with two triangles.

Remark 1.3.12 ((3.3.14)) *For approximating the Dirichlet problem with zero boundary conditions, we use a finite-dimensional space of piecewise polynomial functions satisfying the boundary conditions given by*

$$V_T = \{\mathcal{I}_T f : f \in C^m(\Omega), f|_{\partial\Omega} = 0\}$$

on each triangulation T. This will be discussed further in Chapter 5.

Definition 1.3.13 ((3.3.15)) *We say that an interpolant has continuity order r (in short, that it is "C^r") if $\mathcal{I}_T f \in C^r(\Omega)$ for all $f \in C^m(\Omega)$. The space, $V_T = \{\mathcal{I}_T f : f \in C^m(\Omega)\}$, is said to be a "$C^r$ finite element space".*

Remark 1.3.14 ((3.3.16)) *A finite element (or collection of elements) that can be used to form a C^r space as above is often called a "C^r element". Not all choices of nodes will always lead to C^r continuity, however. Some sort of regularity must be imposed. For the elements studied so far, the essential point is that they be placed in a coordinate-free way that is symmetric with respect to the midpoint of the edge.*

Proposition 1.3.15 ((3.3.17)) ***The Lagrange and Hermite elements are both C^0 elements, and the Argyris element is C^1.*** *More precisely, given a triangulation, $\mathcal{T}$, of Ω, it is possible to choose edge nodes for the corresponding elements $(K, \mathcal{P}, \mathcal{N})$, $K \in \mathcal{T}$, such that the global interpolant satisfies $\mathcal{I}_T f \in C^r$ ($r = 0$ for Lagrange and Hermite, and $r = 1$ for Argyris) for $f \in C^m$ ($m = 0$ for Lagrange, $m = 1$ for Hermite and $m = 2$ for Argyris). In particular, it is sufficient for each edge $\mathbf{x}\mathbf{x}'$ to have nodes $\xi_i(\mathbf{x}' - \mathbf{x}) + \mathbf{x}$, where $\{\xi_i : i = 1, \ldots, k-1-2m\}$ is fixed and symmetric around $\xi = 1/2$. Moreover, under these hypotheses, $\mathcal{I}_T f \in W^{r+1}_\infty$.*

Proof. 证明不重要*It is sufficient to show that the stated continuity holds across each edge. Let $T_i, i = 1, 2$, denote two triangles sharing an edge, e. Since we assumed that the edge nodes were chosen symmetrically* *and in a coordinate-free way, we know that the edge nodes on e for the elements on both T_1 and T_2 are at the same location in space. Let $w := \mathcal{I}_{T_1} f - \mathcal{I}_{T_2} f$, where we view both polynomials, $\mathcal{I}_{T_i} f$ to be defined everywhere by extension outside T_i as polynomials. Then w is a polynomial of degree k and its restriction to the edge e has one-dimensional Lagrange, Hermite or Argyris nodes equal to zero. Thus, $w|_e$ must vanish. Hence, the interpolant is continuous across each edge.*

Lipschitz continuity of $\mathcal{I}_T f$ follows by showing that it has weak derivatives of order $r+1$ given by

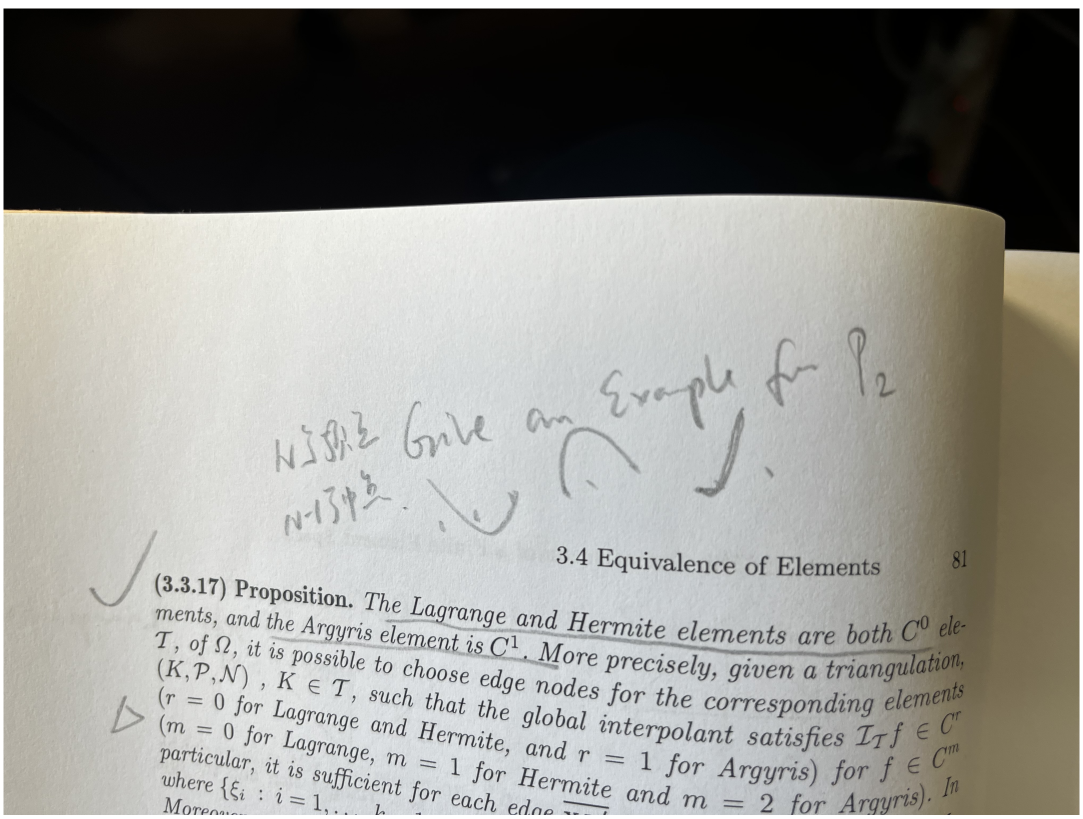
$$(D^\alpha_{(w)} \mathcal{I}_T f)|_T = D^\alpha \mathcal{I}_T f \quad \forall T \in T, |\alpha| \le r+1.$$

The latter is certainly in L^∞. The verification that this is the weak derivative follows from

$$\begin{aligned}\int_\Omega (D^\alpha \phi)(\mathcal{I}_T f)\, dx &= \sum_{T \in T} \int_T (D^\alpha \phi)(\mathcal{I}_T f)\, dx \\ &= \sum_{T \in T} (-1)^{|\alpha|} \int_T \phi (D^\alpha \mathcal{I}_T f)\, dx \\ &= (-1)^{|\alpha|} \int_\Omega \phi \sum_{T \in T} \chi_T (D^\alpha \mathcal{I}_T f)\, dx,\end{aligned}$$

where χ_T denotes the characteristic function of T. The second equality holds because all boundary terms cancel due to the continuity properties of the interpolant. ■

Example 1.3.16 为什么$\mathcal{P}_2$ *Lagrange*元也只有C^0？原因如下：



3.4 Equivalence of Elements 81

(3.3.17) Proposition. *The Lagrange and Hermite elements are both C^0 elements, and the Argyris element is C^1. More precisely, given a triangulation $\mathcal{T}$, of Ω, it is possible to choose edge nodes for the corresponding elements $(K, \mathcal{P}, \mathcal{N})$, $K \in \mathcal{T}$, such that the global interpolant satisfies $\mathcal{I}_\mathcal{T} f \in C^r$ ($r = 0$ for Lagrange and Hermite, and $r = 1$ for Argyris) for $f \in C^m$ ($m = 0$ for Lagrange, $m = 1$ for Hermite and $m = 2$ for Argyris). In particular, it is sufficient for each edge ... where $\{\xi_i : i = 1, \dots$ Moreo...*

Figure 1.10: $\mathcal{P}_2$ Lagrange element.

1.4 3.4 Equivalence of Elements

In the application of the global interpolant, it is essential that we find a uniform bound (independent of $T \in \mathcal{T}$) for the norm of the local interpolation operator $\mathcal{I}_T$. Therefore, we want to compare the local interpolation operators on different elements. The following notions of equivalence are useful for this purpose (cf. Ciarlet & Raviart 1972a).

Definition 1.4.1 ((3.4.1)) *Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element and let $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ (A nonsingular) be an affine map. The finite element $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ is **affine equivalent** to $(K, \mathcal{P}, \mathcal{N})$ if*

(i) $F(K) = \hat{K}$,

(ii) $F^*\hat{\mathcal{P}} = \mathcal{P}$ *and*

(iii) $F_*\mathcal{N} = \hat{\mathcal{N}}$.

We write $\boxed{(K, \mathcal{P}, \mathcal{N}) \underset{F}{\cong} (\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})}$ *if they are affine equivalent.*

Remark 1.4.2 ((3.4.2)) *Recall that the **pull-back** F^* is defined by $F^*(\hat{f}) := \hat{f} \circ F$ and the **push-forward** F_* is defined by $(F_* N)(\hat{f}) := N(F^*(\hat{f}))$.*

Proposition 1.4.3 ((3.4.3)) *Affine equivalence is an equivalence relation.*
Proof. *See exercise 3.x.4.* □ ■

Example 1.4.4 ((3.4.4))

(i) Let K be any triangle, $\mathcal{P} = \mathcal{P}_1$, $\mathcal{N} = \{$evaluation at vertices of $K\}$. All such elements $(K, \mathcal{P}, \mathcal{N})$ are affine equivalent.

(ii) Let K be any triangle, $\mathcal{P} = \mathcal{P}_2$, $\mathcal{N} = \{$evaluation at vertices and edge midpoints$\}$. All such elements are affine equivalent.

(iii) Let $\mathcal{P} = \mathcal{P}_2$. In Fig. 3.14, $(T_1, \mathcal{P}, \mathcal{N}_1)$ and $(T_2, \mathcal{P}, \mathcal{N}_2)$ are not affine equivalent, but the finite elements $(T_1, \mathcal{P}, \mathcal{N}_1)$ and $(T_3, \mathcal{P}, \mathcal{N}_3)$ are affine equivalent.

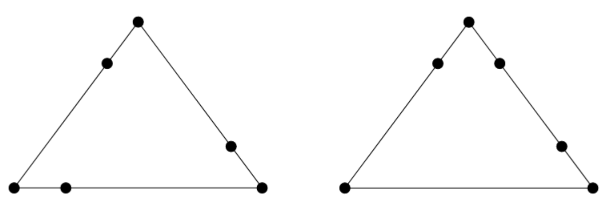


Fig. 3.14. inequivalent quadratic elements: noda placement incompatibility

Figure 1.11: **Fig. 3.14.** inequivalent quadratic elements: nodal placement incompatibility.

*(iv) Let $\mathcal{P} = \mathcal{P}_3$. The elements $(T_1, \mathcal{P}, \mathcal{N}_1)$ and $(T_2, \mathcal{P}, \mathcal{N}_2)$ depicted in Fig. 3.15 are **NOT** affine equivalent since the directional derivatives differ.*

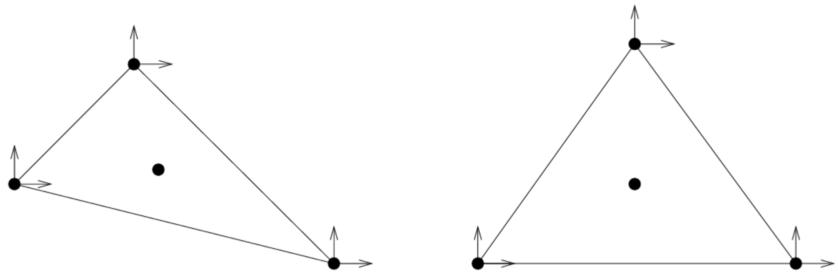


Fig. 3.15. inequivalent cubic Hermite elements: direction incompatibility

Figure 1.12: **Fig. 3.15.** inequivalent cubic Hermite elements: direction incompatibility.

*(v) Let $\mathcal{P} = \mathcal{P}_3$. Then the elements $(T_1, \mathcal{P}, \mathcal{N}_1)$ and $(T_2, \mathcal{P}, \mathcal{N}_2)$ depicted in Fig. 3.16 are **NOT** affine equivalent since the strength of the directional derivatives (indicated by the length of the arrows) differ.*

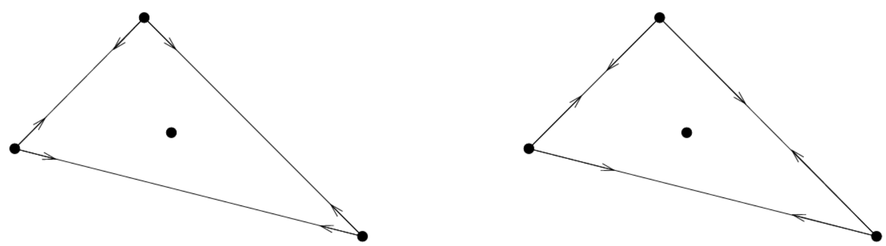


Fig. 3.16. inequivalent cubic Hermite elements: derivative strength incompatibility

Figure 1.13: **Fig. 3.16.** inequivalent cubic Hermite elements: derivative strength incompatibility.

Proposition 1.4.5 ((3.4.5)) *There exist nodal placements such that all Lagrange elements of a given degree are affine equivalent.*

Proof. We pick nodes using barycentric coordinates, (b_1, b_2, b_3), for each triangle. The i-th barycentric coordinate of a point (x, y) can be defined simply as the value of the i-th linear Lagrange basis function at that point ($b_i(x, y) := \phi_i(x, y)$). Thus, each barycentric coordinate is naturally associated with a given vertex; it is equal to the proportional distance of the point from the opposite edge. Note that the barycentric coordinates sum to one (since this yields the interpolant of the constant, 1). Thus, the mapping $(x, y) \to \mathbf{b}(x, y)$ maps the triangle (invertibly) to a subset of $\{\mathbf{b} \in [0, 1]^3 : b_1 + b_2 + b_3 = 1\}$.

For degree k Lagrange elements, pick nodes at the points whose barycentric coordinates are

$$\left(\frac{i}{k}, \frac{j}{k}, \frac{l}{k}\right)$$

where $0 \leq i, j, l \leq k$ and $i + j + l = k$.

■

Definition 1.4.6 ((3.4.6)) *The finite elements $(K, \mathcal{P}, \mathcal{N})$ and $(K, \mathcal{P}, \tilde{\mathcal{N}})$ are interpolation equivalent if*

$$\mathcal{I}_{\mathcal{N}} f = \mathcal{I}_{\tilde{\mathcal{N}}} f \quad \forall f \text{ sufficiently smooth},$$

where $\mathcal{I}_{\mathcal{N}}$ (resp. $\mathcal{I}_{\tilde{\mathcal{N}}}$) is defined by the right-hand side of (3.3.2) with $N_i \in \mathcal{N}$ (resp. $N_i \in \tilde{\mathcal{N}}$). We write $(K, \mathcal{P}, \mathcal{N}) \underset{\mathcal{I}}{\cong} (K, \mathcal{P}, \tilde{\mathcal{N}})$.

Proposition 1.4.7 ((3.4.7)) *Suppose $(K, \mathcal{P}, \mathcal{N})$ and $(K, \mathcal{P}, \widetilde{\mathcal{N}})$ are finite elements. Every nodal variable in $\mathcal{N}$ is a linear combination of nodal variables in $\widetilde{\mathcal{N}}$ (when viewed as a subset of $C^m(K)'$) if and only if $(K, \mathcal{P}, \mathcal{N}) \cong_{\mathcal{I}} (K, \mathcal{P}, \widetilde{\mathcal{N}})$.*

Proof. *(only if) We must show that $\mathcal{I}_{\mathcal{N}} f = \mathcal{I}_{\widetilde{\mathcal{N}}} f \ \forall f \in C^m(K)$. For $N_i \in \mathcal{N}$, we can write $N_i = \sum_{j=1}^k c_j \widetilde{N}_j$ since every nodal variable in $\mathcal{N}$ is a linear combination of nodal variables in $\widetilde{\mathcal{N}}$. Therefore,*

$$
\begin{aligned}
N_i(\mathcal{I}_{\widetilde{\mathcal{N}}} f) &= \left(\sum_{j=1}^k c_j \widetilde{N}_j \right) (\mathcal{I}_{\widetilde{\mathcal{N}}} f) \\
&= \sum_{j=1}^k c_j \widetilde{N}_j (\mathcal{I}_{\widetilde{\mathcal{N}}} f) \\
&= \sum_{j=1}^k c_j \widetilde{N}_j (f) \\
&= N_i(f).
\end{aligned}
$$

The converse is left to the reader in exercise 3.x.26. □ ■

Example 1.4.8 ((3.4.8)) *The Hermite elements in Fig. 3.7 (and 3.15–16) are interpolation equivalent (exercise 3.x.29) while they are not affine equivalent.*

Definition 1.4.9 ((3.4.9)) *If $(K, \mathcal{P}, \mathcal{N})$ is a finite element that is affine equivalent to $(\tilde{K}, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ and $(\tilde{K}, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ is interpolation equivalent to $(\tilde{K}, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$, then we say that $(K, \mathcal{P}, \mathcal{N})$ is affine-interpolation equivalent to $(\tilde{K}, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$.*

Example 1.4.10 ((3.4.10))

(i) All affine equivalent elements (e.g., Lagrange elements with appropriate choices for the edge and interior nodes as described in Proposition 3.4.5) are affine-interpolation equivalent.

(ii) The Hermite elements with appropriate choices for the edge and interior nodes are affine-interpolation equivalent.

(iii) The Argyris elements are not affine-interpolation equivalent (Ciarlet 1978). ???

The following is an immediate consequence of the definitions.

Proposition 1.4.11 ((3.4.11)) *If $(K, \mathcal{P}, \mathcal{N})$ is affine-interpolation equivalent to $(\tilde{K}, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$ then $\mathcal{I} \circ F^* = F^* \circ \tilde{\mathcal{I}}$ where F is the affine mapping $K \to \tilde{K}$.*