MATH2103: Lecture Note on Numerical Solution of Partial Differential Equations

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Contents

Chapter 1

Variational Formulation of Elliptic Boundary Value Problems

This chapter is devoted to the functional analysis tools required for developing the variational formulation of differential equations. It begins with an introduction to Hilbert spaces, including only material that is essential to later developments. The goal of the chapter is to provide a framework in which **existence and uniqueness of solutions to variational problems** may be established.

1.1 2.1 Inner-Product Spaces

Definition 1.1.1 ((2.1.1)) *A* ***bilinear form****, $b(\cdot,\cdot)$, on a linear space V is a mapping $b : V \times V \longrightarrow \mathbb{R}$ such that each of the maps $v \mapsto b(v,w)$ and $w \mapsto b(v,w)$ is a linear form on V. It is* ***symmetric*** *if $b(v,w) = b(w,v)$ for all $v, w \in V$. A (real)* ***inner product****, denoted by $(\cdot,\cdot)$, is a symmetric bilinear form on a linear space V that satisfies*

$$\begin{array}{llcl} (a) & (v,v) & \geq & 0 \quad \forall v \in V \\ (b) & (v,v) & = & 0 \Longleftrightarrow v = 0. \end{array}$$

Definition 1.1.2 ((2.1.2)) *A linear space V together with an inner product defined on it is called an* ***inner-product space*** *and is denoted by* $\boxed{(V,(\cdot,\cdot))}$.

Example 1.1.3 ((2.1.3)) *The following are examples of inner-product spaces.*

$$\begin{array}{ll} (i) & V = \mathbb{R}^n,\ (x,y) := \sum_{i=1}^{n} x_i y_i \\ (ii) & V = L^2(\Omega),\ \Omega \subseteq \mathbb{R}^n,\ (u,v)_{L^2(\Omega)} := \int_\Omega u(x)v(x)\,dx \\ (iii) & V = W_2^k(\Omega),\ \Omega \subseteq \mathbb{R}^n,\ (u,v)_k := \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} \end{array}$$

Notation. The inner-product space (iii) is often denoted by $H^k(\Omega)$. Thus, $H^k(\Omega) = W_2^k(\Omega)$.

Theorem 1.1.4 ((2.1.4) The Schwarz Inequality) *If $(V, (\cdot, \cdot))$ is an inner-product space, then*

$$|(u, v)| \le (u, u)^{1/2}(v, v)^{1/2}. \tag{2.1.5}$$

The equality holds if and only if u and v are linearly dependent.
Proof. *For $t \in \mathbb{R}$,*

$$0 \le (u - tv, u - tv) = (u, u) - 2t(u, v) + t^2(v, v). \tag{2.1.6}$$

If $(v, v) = 0$, then $(u, u) - 2t(u, v) \ge 0$ for all $t \in \mathbb{R}$, which forces $(u, v) = 0$, so the inequality holds trivially. Thus, suppose $(v, v) \ne 0$. Substituting $t = (u, v)/(v, v)$ into this inequality, we obtain

$$0 \le (u, u) - |(u, v)|^2/(v, v),$$

which is equivalent to the Schwarz inequality (2.1.5). Note that we did not use part (b) of Definition 2.1.1 to prove (2.1.5).

If u and v are linearly dependent, one can easily see that equality holds in the inequality (2.1.5).

Conversely, assume that equality holds. If $v = 0$, then u and v are linearly dependent. If $v \ne 0$, take $\lambda = (u, v)/(v, v)$. It follows that $(u - \lambda v, u - \lambda v) = 0$, and property (b) of Definition 2.1.1 implies that $u - \lambda v = 0$, i.e., u and v are linearly dependent. ■

Remark 1.1.5 ((2.1.8)) *The Schwarz inequality (2.1.5) was proved without using property (b) of the inner product. An example where this might be useful is*

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{on } H^1(\Omega).$$

Thus, we have proved that

$$a(u, v)^2 \le a(u, u)\, a(v, v) \quad \text{for all } u, v \in H^1(\Omega),$$

even though $a(\cdot, \cdot)$ ***is not an inner product on*** $H^1(\Omega)$ *since $a(u, u) = 0 \Longrightarrow u = C$.*

Proposition 1.1.6 ((2.1.9)) $\|v\| := \sqrt{(v, v)}$ *defines a norm in the inner-product space $(V, (\cdot, \cdot))$.*
Proof. *One can easily show that $\|\alpha v\| = |\alpha|\|v\|$, $\|v\| \ge 0$, and $\|v\| = 0 \Longleftrightarrow v = 0$. It remains to prove the triangle inequality:*

$$\begin{aligned}
\|u + v\|^2 &= (u + v, u + v) \\
&= (u, u) + 2(u, v) + (v, v) \\
&= \|u\|^2 + 2(u, v) + \|v\|^2 \\
&\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \textit{(by Schwarz' inequality (2.1.5))} \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

Therefore, $\|u + v\| \le \|u\| + \|v\|$. ■

1.2 2.2 Hilbert Spaces

Proposition 2.1.9 says that, given an inner-product space $(V, (\cdot, \cdot))$, there is an associated norm defined on V, namely $\|v\| = \sqrt{(v, v)}$. Thus, an inner-product space can be made into a normed linear space.

Definition 1.2.1 ((2.2.1)) *Let $(V,(\cdot,\cdot))$ be an inner-product space. If the associated normed linear space $(V,\|\cdot\|)$ is complete, then $(V,(\cdot,\cdot))$ is called a* ***Hilbert space****.*

Example 1.2.2 ((2.2.2)) *The examples (i) - (iii) of (2.1.3) are all Hilbert spaces. In particular, the norm associated with the inner product $(\cdot,\cdot)_k$ on $W_2^k(\Omega)$ is the same as the norm $\|\cdot\|_{W_2^k(\Omega)}$ defined in Chapter 1 where $W_2^k(\Omega)$ was shown to be complete.*

Definition 1.2.3 ((2.2.3)) *Let H be a Hilbert space and $S \subset H$ be a linear subset that is closed in H. (Recall that S* ***linear*** *means that $u, v \in S, \alpha \in \mathbb{R} \Longrightarrow u + \alpha v \in S$.) Then S is called a* ***subspace of*** *H.*

Proposition 1.2.4 ((2.2.4)) *If S is a subspace of H, then $(S,(\cdot,\cdot))$ is also a Hilbert space. 子空间也是Hilbert.*

Proof. $(S,\|\cdot\|)$ is complete because S is closed in H under the norm $\|\cdot\|$. ■

Example 1.2.5 ((2.2.5) Examples of subspaces of Hilbert spaces.)

(i) *H and $\{0\}$ are the obvious extreme cases. More interesting ones follow.*

(ii) *Let $T : H \longrightarrow K$ be a continuous linear map of H into another linear space. Then $\boxed{\ker T}$ is a subspace (see exercise 2.x.1).*

(iii) *Let $x \in H$ and define $x^\perp := \{v \in H : (v, x) = 0\}$. Then $x^\perp$ is a subspace of H. To see this, note that $x^\perp = \ker L_x$, where L_x is the linear functional*

$$L_x : v \mapsto (v, x). \tag{2.2.6}$$

By the Schwarz inequality (2.1.5),

$$|L_x(v)| \leq \|x\|\|v\|$$

implying that L_x is bounded and therefore continuous. This proves that $x^\perp$ is a subspace of H in view of the previous example.

Note. *The overall objective of the next section is to prove that all $L \in H'$ 对偶 are of the form L_x for $x \in H$, when H is a Hilbert space.*

(iv) *Let $M \subset H$ be a subset and define*

$$M^\perp := \{v \in H : (x, v) = 0 \, \forall x \in M\}.$$

Note that

$$M^\perp = \bigcap_{x \in M} x^\perp$$

and each $x^\perp$ is a (closed) subspace of H. Thus, $M^\perp$ is a subspace of H.

Proposition 1.2.6 ((2.2.7)) *Let H be a Hilbert space.*

(1) *For any subsets $M, N \subset H, M \subset N \implies N^\perp \subset M^\perp$.*

(2) For any subset M of H containing zero,

$$M \cap M^{\perp} = \{0\}.$$

(3) $\{0\}^{\perp} = H$.

(4) $H^{\perp} = \{0\}$.

Proof. For (2): Let $x \in M \cap M^{\perp}$. Then $x \in M \implies M^{\perp} \subset x^{\perp}$ and so

$$x \in M^{\perp} \implies x \in x^{\perp} \iff (x,x) = 0 \iff x = 0.$$

For (4): Since $H^{\perp} \subset H$, (2) implies that

$$H^{\perp} = H \cap H^{\perp} = \{0\}.$$

Parts (1) and (3) are left to the reader in exercise 2.x.3. ■

Theorem 1.2.7 ((2.2.8) Parallelogram Law) *Let $\|\cdot\|$ be the norm associated with the inner product $(\cdot,\cdot)$ on H. We have*

$$\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right). \tag{2.2.9}$$

Proof. *A straight-forward calculation; see exercise 2.x.4.* ■

1.3 2.3 Projections onto Subspaces

The following result establishes an essential geometric fact about Hilbert spaces.

Proposition 1.3.1 ((2.3.1)) *Let M be a subspace of the Hilbert space H. Let $v \in H \setminus M$ and define $\delta := \inf\{\|v-w\| : w \in M\}$. (Note that $\delta > 0$ since M **is closed in** H. closed非常重要保证M的极限还是在M中) Then there exists $w_0 \in M$ such that*

(i) $\|v - w_0\| = \delta$, i.e., there exists a closest point $w_0 \in M$ to v, and

(ii) $v - w_0 \in M^{\perp}$.

Proof. (i) Let $\{w_n\}$ be a minimizing sequence:

$$\lim_{n\to\infty} \|v - w_n\| = \delta.$$

We now show that $\{w_n\}$ is a Cauchy sequence. By the parallelogram law,

$$\|(w_n - v) + (w_m - v)\|^2 + \|(w_n - v) - (w_m - v)\|^2 = 2(\|w_n - v\|^2 + \|w_m - v\|^2),$$

i.e.,

$$0 \le \|w_n - w_m\|^2 = 2(\|w_n - v\|^2 + \|w_m - v\|^2) - 4\left\|\frac{1}{2}(w_n + w_m) - v\right\|^2.$$

Since $\frac{1}{2}(w_n + w_m) \in M$, we have

$$\left\|\frac{1}{2}(w_n + w_m) - v\right\| \ge \delta(\text{下确界})$$

by the definition of δ. Thus,

$$0 \leq \|w_n - w_m\|^2 \leq 2(\|w_n - v\|^2 + \|w_m - v\|^2) - 4\delta^2.$$

Letting m, n tend towards infinity, we have

$$2(\|w_n - v\|^2 + \|w_m - v\|^2) - 4\delta^2 \to 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.$$

Therefore, $\|w_n - w_m\|^2 \to 0$, proving that $\{w_n\}$ is Cauchy. Thus, there exists $w_0 \in M = M$ such that $w_n \to w_0$ ($\|\cdot\|$完备). Continuity of the norm implies that $\|v - w_0\| = \delta$.

(这部分证明有趣重要) (ii) Let $z = v - w_0$, so that $\|z\| = \delta$. We will prove that $z \perp M$. Let $w \in M$ and $t \in \mathbb{R}$. Then $w_0 + tw \in M$ implies that $\|z - tw\|^2 = \|v - (w_0 + tw)\|^2$ has an absolute minimum at $t = 0$. Therefore,

$$0 = \frac{d}{dt}\|z - tw\|^2|_{t=0} = -2(z, w).$$

This implies that, for all $w \in M$,

$$(v - w_0, w) = (z, w) = 0.$$

Since $w \in M$ was arbitrary, this implies $v - w_0 \in M^{\perp}$. ∎

Proposition 2.3.1 says that, given a subspace M of H and $v \in H$, we can write $v = w_0 + w_1$, where $w_0 \in M$ and $w_1 (= v - w_0) \in M^{\perp}$. Let us show that this decomposition of an element $v \in H$ is **unique**. In fact, from

$$w_0 + w_1 = v = z_0 + z_1, \quad w_0, z_0 \in M, \quad w_1, z_1 \in M^{\perp},$$

we obtain

$$M \ni w_0 - z_0 = -(w_1 - z_1) \in M^{\perp}.$$

Since $M \cap M^{\perp} = \{0\}$, we have $w_0 = z_0$ and $w_1 = z_1$. This shows that the decomposition is unique. Therefore, we can define the following operators:

$$P_M : H \longrightarrow M, \quad P_M^{\perp} : H \longrightarrow M^{\perp} \tag{2.3.2}$$

where the respective definitions of P_M and $P_M^{\perp}$ are given by

$$P_M v = \begin{cases} v & \text{if } v \in M, \\ w_0 & \text{if } v \in H \setminus M; \end{cases} \tag{2.3.3}$$

$$P_M^{\perp} v = \begin{cases} 0 & \text{if } v \in M, \\ v - w_0 & \text{if } v \in H \setminus M. \end{cases} \tag{2.3.4}$$

The uniqueness of the decomposition implies that $P_M^{\perp} = P_{M^{\perp}}$ (see exercise 2.x.5) so we need no longer be careful about where we put the “$\perp$”. Summarizing the above observations, we state the following:

Proposition 1.3.2 ((2.3.5)) *Given a subspace M of H and $v \in H$, there is a unique decomposition*

$$v = P_M v + P_{M^{\perp}} v, \tag{2.3.6}$$

where $P_M : H \longrightarrow M$ and $P_{M^\perp} : H \longrightarrow M^\perp$. In other words,

$$H = M \oplus M^\perp. \tag{2.3.7}$$

Remark 1.3.3 ((2.3.8)) *The operators P_M and $P_{M^\perp}$ defined above are linear operators. To see this, note from the above proposition that*

$$\alpha v_1 + \beta v_2 = P_M(\alpha v_1 + \beta v_2) + P_{M^\perp}(\alpha v_1 + \beta v_2),$$

where

$$v_1 = P_M v_1 + P_{M^\perp} v_1 \quad \text{and} \quad v_2 = P_M v_2 + P_{M^\perp} v_2.$$

That is,

$$P_M(\alpha v_1 + \beta v_2) + P_{M^\perp}(\alpha v_1 + \beta v_2) = \alpha v_1 + \beta v_2 = (\alpha P_M v_1 + \beta P_M v_2) + (\alpha P_{M^\perp} v_1 + \beta P_{M^\perp} v_2).$$

Uniqueness of decomposition of $\alpha v_1 + \beta v_2$ and the definitions of P_M and $P_{M^\perp}$ imply that

$$P_M(\alpha v_1 + \beta v_2) = \alpha P_M v_1 + \beta P_M v_2,$$

and

$$P_{M^\perp}(\alpha v_1 + \beta v_2) = \alpha P_{M^\perp} v_1 + \beta P_{M^\perp} v_2,$$

i.e., P_M and $P_{M^\perp}$ are linear.

Definition 1.3.4 ((2.3.9)) *An operator P on a linear space V is a* ***projection*** *if $P^2 = P$, i.e., $Pz = z$ for all z in the image of P.*

Remark 1.3.5 ((2.3.10)) *The fact that P_M is a projection follows from its definition. That $P_M^\perp$ is also follows from the observation that $P_M^\perp = P_{M^\perp}$.*

1.4 2.4 Riesz Representation Theorem

Given $u \in H$, recall that a continuous linear functional L_u can be defined on H by

$$L_u(v) = (u, v). \tag{2.4.1}$$

The following theorem proves that the converse is also true.

Theorem 1.4.1 ((2.4.2) Riesz Representation Theorem) *Any continuous linear functional L on a Hilbert space H can be represented uniquely as*

$$L(v) = (u, v) \tag{2.4.3}$$

for some $u \in H$. Furthermore, we have

$$\|L\|_{H'} = \|u\|_H. \tag{2.4.4}$$

Proof. Uniqueness follows from the nondegeneracy of the inner product. For if u_1 and u_2 were two such solutions, we would have

$$\begin{aligned}0 &= L(u_1 - u_2) - L(u_1 - u_2)\\ &= (u_1, u_1 - u_2) - (u_2, u_1 - u_2)\\ &= (u_1 - u_2, u_1 - u_2)\end{aligned}$$

which implies $u_1 = u_2$. Now we prove existence.

Define $M := \{v \in H : L(v) = 0\}$. In view of Example 2.2.5.ii, M is a subspace of H. Therefore, $H = M \oplus M^{\perp}$ by Proposition 2.3.5.

Case (1): $M^{\perp} = \{0\}$.

Thus, in this case $M = H$, implying that $L \equiv 0$. So take $u = 0$.

Case (2): $M^{\perp} \neq \{0\}$.

Pick $\boxed{z \in M^{\perp}}$, $z \neq 0$. Then $L(z) \neq 0$. (Otherwise, $z \in M$, which implies that $z \in M^{\perp} \cap M = \{0\}$.) For $v \in H$ and $\beta = L(v)/L(z)$ we have

$$L(v - \beta z) = L(v) - \beta L(z) = 0,$$

i.e.

$$v - \beta z \in M.$$

Thus, $v - \beta z = P_M v$ and $\beta z = P_{M^{\perp}} v$. (not important comment: In particular, if $v \in M^{\perp}$, then $v = \beta z$ (that is, $v - \beta z = 0$), which proves that $M^{\perp}$ is one-dimensional.)

Now choose

$$u := \frac{L(z)}{\|z\|_H^2} z. \tag{2.4.5}$$

Note that $u \in M^{\perp}$. We have

$$\begin{aligned}(u, v) &= (u, (v - \beta z) + \beta z)\\ &= (u, v - \beta z) + (u, \beta z)\\ &= (u, \beta z) \quad (u \in M^{\perp},\ v - \beta z \in M)\\ &= \beta \frac{L(z)}{\|z\|_H^2}(z, z) \quad \text{(definition of } u)\\ &= \beta L(z)\\ &= L(v). \quad \text{(definition of } \beta)\end{aligned}$$

Thus, $u := (L(z)/\|z\|^2)\, z$ is the desired element of H.

It remains to prove that $\|L\|_{H'} = \|u\|_H$. Let us first observe that

$$\|u\|_H = \frac{|L(z)|}{\|z\|_H}$$

from (2.4.5). Now, according to the definition (1.7.2) of the dual norm,

$$\begin{aligned}
\|L\|_{H'} &= \sup_{0\neq v\in H}\frac{|L(v)|}{\|v\|_H} \quad \text{(by 1.7.2)}\\
&= \sup_{0\neq v\in H}\frac{|(u,v)|}{\|v\|_H} \quad \text{(by 2.4.3)}\\
&\leq \|u\|_H \quad \text{(Schwarz' inequality 2.1.5)}\\
&= \frac{|L(z)|}{\|z\|_H} \quad \text{(by 2.4.5)}\\
&\leq \|L\|_{H'} \quad \text{(by 1.7.2)}
\end{aligned}$$

Therefore, $\|u\|_H = \|L\|_{H'}$. ■

Remark 1.4.2 ((2.4.6)) *According to the Riesz Representation Theorem, there is a natural isometry between H and H' ($u \in H \iff L_u \in H'$). For this reason, H and H' are often identified. For example, we can write $W_2^m(\Omega) \cong W_2^{-m}(\Omega)$ (although they are completely different Hilbert spaces). We will use τ to represent the **isometry** from H' onto H.*

1.5 2.5 Formulation of Symmetric Variational Problems

The purpose of the rest of this chapter is to apply the abstract Hilbert space theory developed in the previous sections to get existence and uniqueness results for variational formulations of boundary value problems.

Example 1.5.1 ((2.5.1)) *Recall from Examples 2.1.3.iii and 2.2.2 that $H^1(0,1) = W_2^1(0,1)$ is a Hilbert space under the inner product*

$$(u,v)_{H^1} = \int_0^1 uv\,dx + \int_0^1 u'v'\,dx.$$

In Chapter 0, we defined $\boxed{V = \{v \in H^1(0,1) : v(0) = 0\}}$. To see that V is a subspace of $H^1(0,1)$, let $\delta_0 : H^1(0,1) \longrightarrow \mathbb{R}$ by $\delta_0(v) = v(0)$. From Sobolev's inequality (1.4.6), δ_0 is a bounded linear functional on H^1, so it is continuous. Hence, $V = \delta_0^{-1}\{0\}$ is closed in H^1. We also defined $a(v,w) = \int_0^1 v'w'\,dx$. Note that $a(\cdot,\cdot)$ is a symmetric bilinear form on $H^1(0,1)$, but it is not an inner product on H^1 since $a(1,1) = 0$. However, it does satisfy the coercivity property (1.5.1) on V. In view of the following, this shows that the variational problem in Chapter 0 is naturally expressed in a Hilbert-space setting.

Definition 1.5.2 ((2.5.2)) 非常重要的定义 *A bilinear form $a(\cdot,\cdot)$ on a normed linear space H is said to be **bounded** (or **continuous**) if $\exists C < \infty$ such that*

$$|a(v,w)| \leq C\|v\|_H\|w\|_H \quad \forall v,w \in H$$

*and **coercive** on $V \subset H$ if $\exists \alpha > 0$ such that*

$$a(v,v) \geq \alpha\|v\|_H^2 \quad \forall v \in V.$$

Proposition 1.5.3 ((2.5.3)) *Let H be a Hilbert space, and suppose $a(\cdot,\cdot)$ is a symmetric bilinear form that is continuous on H and coercive on a subspace V of H. Then $(V, a(\cdot,\cdot))$ **is a Hilbert space.***

Proof. 不重要的证明 An immediate consequence of the coercivity of $a(\cdot,\cdot)$ is that if $v \in V$ and $a(v,v)=0$, then $v \equiv 0$. Hence, $a(\cdot,\cdot)$ is an inner product on V.

Now let $\|v\|_E = \sqrt{a(v,v)}$, and suppose that $\{v_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_E)$. By coercivity, $\{v_n\}$ is also Cauchy in $(H, \|\cdot\|_H)$. Since H is complete, $\exists v \in H$ such that $v_n \to v$ in the $\|\cdot\|_H$ norm. Since V is closed in H, $v \in V$. Now, $\|v - v_n\|_E \le \sqrt{C}\|v - v_n\|_H$ since $a(\cdot,\cdot)$ is bounded. Hence, $\{v_n\} \to v$ in the $\|\cdot\|_E$ norm, so $(V, \|\cdot\|_E)$ is complete. ■

In general, a symmetric variational problem is posed as follows. Suppose that the following three conditions are valid:

$$\begin{cases} (1) \quad (H,(\cdot,\cdot)) \text{ is a Hilbert space.} \\ (2) \quad V \text{ is a (closed) subspace of } H. \\ (3) \quad a(\cdot,\cdot) \text{ is a } \textbf{bounded, symmetric bilinear} \text{ form that is } \textbf{coercive} \text{ on } V. \end{cases} \tag{2.5.4}$$

Then the **symmetric variational problem** is the following.
(2.5.5) Given $F \in V'$, find $u \in V$ such that $a(u,v) = F(v)\ \forall v \in V$.

Theorem 1.5.4 ((2.5.6)) *Suppose that conditions (1) –(3) of (2.5.4) hold. Then there exists a unique $u \in V$ solving (2.5.5).*

Proof. Proposition 2.5.3 implies that $a(\cdot,\cdot)$ is an inner product on V and that $(V, a(\cdot,\cdot))$ is a Hilbert space. Apply the Riesz Representation Theorem. ■

重要The **(Ritz-Galerkin) Approximation Problem** is the following.

Given a finite-dimensional subspace $V_h \subset V$ and $F \in V'$, find $u_h \in V_h$ such that

$$a(u_h, v) = F(v) \quad \forall v \in V_h. \tag{2.5.7}$$

Theorem 1.5.5 ((2.5.8)) *Under the conditions (2.5.4), there exists a unique u_h that solves (2.5.7).*

Proof. $\boxed{(V_h, a(\cdot,\cdot))}$ **is a Hilbert space in its own right (V_h is closed)**, and $F|_{V_h} \in V_h'$. Apply the Riesz Representation Theorem. ■

Error estimates for $u - u_h$ are a consequence of the following relationship.

Proposition 1.5.6 ((2.5.9) Fundamental Galerkin Orthogonality) *Let u and u_h be solutions to (2.5.5) and (2.5.7) respectively. Then*

$$a(u - u_h, v) = 0 \quad \forall v \in V_h.$$

Proof. Subtract the two equations

$$a(u,v) = F(v) \quad \forall v \in V$$

$$a(u_h,v) = F(v) \quad \forall v \in V_h.$$

■

Corollary 1.5.7 ((2.5.10)) $\|u - u_h\|_E = \min_{v \in V_h} \|u - v\|_E$.
Proof. *Same as (0.3.3).* ■

Remark 1.5.8 ((2.5.11) The Ritz Method) *In the symmetric case, u_h minimizes the quadratic functional*

$$Q(v) = a(v, v) - 2F(v)$$

over all $v \in V_h$ (see exercise 2.x.6).

非常重要Note that (2.5.10) and (2.5.11) are valid only in the symmetric case.

1.6 2.6 Formulation of Nonsymmetric Variational Problems

A nonsymmetric variational problem is posed as follows. Suppose that the following five conditions are valid:

$$\begin{cases} (1) \quad (H, (\cdot,\cdot)) \text{ is a Hilbert space.} \\ (2) \quad V \text{ is a (closed) subspace of } H. \\ (3) \quad a(\cdot,\cdot) \text{ is a bilinear form on } V, \ \textbf{not necessarily symmetric}. \\ (4) \quad a(\cdot,\cdot) \text{ is continuous (bounded) on } V. \\ (5) \quad a(\cdot,\cdot) \text{ is coercive on } V. \end{cases} \tag{2.6.1}$$

Then the **nonsymmetric variational problem** is the following.

(2.6.2) Given $F \in V'$, find $u \in V$ such that $a(u, v) = F(v)\ \forall v \in V$.

The **(Galerkin) approximation problem** is the following.

Given a finite-dimensional subspace $V_h \subset V$ and $F \in V'$, find $u_h \in V_h$ such that

$$a(u_h, v) = F(v) \quad \forall v \in V_h. \tag{2.6.3}$$

The following questions arise.

1. Do there exist unique solutions u, u_h?

2. What are the error estimates for $u - u_h$?

3. Are there any interesting examples?

An Interesting Example.

Consider the boundary value problem

$$-u'' + u' + u = f \quad \text{on } [0, 1] \quad u'(0) = u'(1) = 0. \tag{2.6.4}$$

One variational formulation for this is: Take

$$V = H^1(0, 1)$$
$$a(u, v) = \int_0^1 (u'v' + u'v + uv)\, dx \tag{2.6.5}$$

$$F(v) = (f, v)$$

and solve the variational equation (2.6.2). Note that $a(\cdot,\cdot)$ is **not symmetric** because of the $u'v$ term.
Proof. To prove $a(\cdot,\cdot)$ is continuous, observe that

$$\begin{aligned}
|a(u,v)| &\le |(u,v)_{H^1}| + \left| \int_0^1 u'v \, dx \right| \\
&\le \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L^2} \quad \text{(Schwarz' inequality 2.1.5)} \\
&\le 2\|u\|_{H^1}\|v\|_{H^1}.
\end{aligned}$$

Therefore, $a(\cdot,\cdot)$ is continuous (take $c_1 = 2$ in the definition).

To prove $a(\cdot,\cdot)$ is coercive, observe that

$$\begin{aligned}
a(v,v) &= \int_0^1 \left(v'^2 + v'v + v^2\right) dx \\
&= \frac{1}{2}\int_0^1 (v'+v)^2 \, dx + \frac{1}{2}\int_0^1 \left(v'^2 + v^2\right) dx \\
&\ge \frac{1}{2}\|v\|_{H^1}^2.
\end{aligned}$$

Therefore, $a(\cdot,\cdot)$ is coercive (take $c_2 = 1/2$ in the definition). ■

不重要 If the above differential equation is changed to

$$-u'' + ku' + u = f, \tag{2.6.6}$$

then the corresponding $a(\cdot,\cdot)$ need not be coercive for large k.

Remark 1.6.1 ((2.6.7)) *If $(H, (\cdot,\cdot))$ is a Hilbert space, V is a subspace of H, and $a(\cdot,\cdot)$ is an inner product on V, then $(V, a(\cdot,\cdot))$ need not be complete if $a(\cdot,\cdot)$ is not coercive. For example, let $H = H^1(0,1)$, $V = H$, $a(v,w) = \int_0^1 vw\,dx = (v,w)_{L^2(0,1)}$. Then $a(\cdot,\cdot)$ is an inner product in V, but convergence in the L^2 norm does not imply convergence in the H^1 norm since $H^1(0,1)$ is dense in $L^2(0,1)$.*

1.7 2.7 The Lax-Milgram Theorem

We would like to prove the existence and uniqueness of the solution of the **(nonsymmetric)** variational problem: Find $u \in V$ such that

$$a(u,v) = F(v) \quad \forall v \in V,$$

where V is a Hilbert space, $F \in V'$ and $a(\cdot,\cdot)$ is a **continuous, coercive bilinear form that is not necessarily symmetric**. The Lax-Milgram Theorem guarantees both existence and uniqueness of the solution to (2.7.1). First we need to prove the following lemma.

Lemma 1.7.1 ((2.7.2) Contraction Mapping Principle) *Given a Banach space V and a mapping $T : V \longrightarrow V$, satisfying*

$$\|Tv_1 - Tv_2\| \le M\|v_1 - v_2\|$$

for all $v_1, v_2 \in V$ and fixed M, $0 \le M < 1$, there exists a unique $u \in V$ such that

$$u = Tu,$$

i.e. ***the contraction mapping T has a unique fixed point u*** **不动点定理**.

Remark 1.7.2 ((2.7.5)) *We actually only need that V is a complete metric space in the lemma.* 不一定是赋范线性

Theorem 1.7.3 ((2.7.7) Lax-Milgram) *Given a Hilbert space $(V, (\cdot,\cdot))$, a continuous, coercive bilinear form $a(\cdot,\cdot)$ and a continuous linear functional $F \in V'$, there exists a unique $u \in V$ such that*

$$a(u, v) = F(v) \quad \forall v \in V,$$

where a might be nonsymmetric.

Theorem 1.7.4 (Lax-Milgram Theorem) *Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that*
(i)

$$|B[u, v]| \leq \alpha \|u\| \|v\| \quad (u, v \in H)$$

and
(ii)

$$\beta \|u\|^2 \leq B[u, u] \quad (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H.

Then there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

Proof. 1. For each fixed element $u \in H$, the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H; whence the Riesz Representation Theorem (§D.3) asserts the existence of a unique element $w \in H$ satisfying

$$B[u, v] = (w, v) \quad (v \in H). \tag{2}$$

Let us write $Au = w$ whenever (2) holds; so that

$$B[u, v] = (Au, v) \quad (u, v \in H). \tag{3}$$

2. We first claim $A : H \to H$ is a bounded linear operator. Indeed if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$, we see for each $v \in H$ that

$$\begin{aligned}
(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \quad \text{by (3)} \\
&= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\
&= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \quad \text{by (3) again} \\
&= (\lambda_1 Au_1 + \lambda_2 Au_2, v).
\end{aligned}$$

This equality obtains for each $v \in H$, and so A is linear. Furthermore

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|.$$

Consequently $\|Au\| \leq \alpha \|u\|$ for all $u \in H$, and so A is bounded.

3. Next we assert

$$\begin{cases} A \text{ is one-to-one, and} \\ R(A), \text{ the range of } A, \text{ is closed in } H. \end{cases} \tag{4}$$

To prove this, let us compute

$$\beta\|u\|^2 \leq B[u,u] = (Au,u) \leq \|Au\|\|u\|.$$

Hence $\beta\|u\| \leq \|Au\|$. This inequality easily implies (4).

4. We demonstrate now

$$R(A) = H. \tag{5}$$

For if not, then, since $R(A)$ is closed, there would exist a nonzero element $w \in H$ with $w \in R(A)^{\perp}$. But this fact in turn implies the contradiction

$$\beta\|w\|^2 \leq B[w,w] = (Aw,w) = 0.$$

5. Next, we observe once more from the Riesz Representation Theorem that

$$\langle f, v\rangle = (w,v) \quad \text{for all } v \in H$$

for some element $w \in H$. We then utilize (4) and (5) to find $u \in H$ satisfying $Au = w$. Then

$$B[u,v] = (Au,v) = (w,v) = \langle f,v\rangle \quad (v \in H),$$

and this is (1).

6. Finally, we show there is at most one element $u \in H$ verifying (1). For if both $B[u,v] = \langle f,v\rangle$ and $B[\tilde{u},v] = \langle f,v\rangle$, then

$$B[u-\tilde{u},v] = 0 \quad (v \in H).$$

We set $v = u - \tilde{u}$ to find

$$\beta\|u-\tilde{u}\|^2 \leq B[u-\tilde{u},u-\tilde{u}] = 0.$$

Thus $u = \tilde{u}$. □

■

Remark 1.7.5 *If the bilinear form $B[\cdot,\cdot]$ is symmetric, that is, if*

$$B[u,v] = B[v,u] \quad (u,v \in H),$$

we can fashion a much simpler proof by noting $((u,v)) := B[u,v]$ is a new inner product on H, to which the Riesz Representation Theorem directly applies. Consequently, the Lax-Milgram Theorem is primarily significant in that it does not require symmetry of $B[\cdot,\cdot]$. □

Remark 1.7.6 ((2.7.11)) *Note that $\|u\|_V \leq (1/\alpha)\|F\|_{V'}$ where α is the coercivity constant (see exercise 2.x.9).*

Corollary 1.7.7 ((2.7.12)) *Under conditions (2.6.1), the variational problem (2.6.2) has a unique solution.*

Proof. Conditions (1) and (2) of (2.6.1) imply that $(V,(\cdot,\cdot))$ is a Hilbert space. Apply the Lax-Milgram Theorem. ■

Corollary 1.7.8 ((2.7.13)) *Under the conditions (2.6.1), the approximation problem (2.6.3) has a unique solution.*

Proof. Since V_h is a (closed) subspace of V, (2.6.1) holds with V replaced by V_h. Apply the previous corollary. ■

Remark 1.7.9 ((2.7.14)) *Note that V_h need not be finite-dimensional for (2.6.3) to be well-posed.*

1.8 Application of Lax-Milgram theorem in Harlim's note

1.8.1 The Ritz method

Proposition 1.8.1 (2.13) *Let the hypotheses of the Lax-Milgram to be valid. Assume that the bilinear form a is* ***symmetric****, then the solution of the variational problem in (2.25) is also the unique solution of the minimization problem,*

$$F(u) = \inf_{\nu \in H} F(\nu), \tag{2.33}$$

where

$$F(\nu) = \frac{1}{2}a(\nu, \nu) - \ell(\nu).$$

Proof. Let u be the Lax-Milgram solution, then for all $\nu \in H$, define $w = \nu - u$ and

$$\begin{aligned} F(\nu) &= \frac{1}{2}a(\nu, \nu) - \ell(\nu) \\ &= \frac{1}{2}a(u, u) + \frac{1}{2}a(u, w) + \frac{1}{2}a(w, u) + \frac{1}{2}a(w, w) - \ell(u) - \ell(w) \\ &= F(u) + a(u, w) - \ell(w) + \frac{1}{2}a(w, w) \\ &\geq F(u) \end{aligned} \tag{2.34}$$

since $a(w, w) \geq 0$, and we have used the symmetry property to remove cross terms. Thus u minimizes F.

Now, we show the converse. Let u be the solution of the minimization problem in (2.33). For any $c > 0$ and $\nu \in H$, $F(u + cv) \geq F(u)$. This is equivalent to,

$$g(c) := \frac{1}{2}a(u, u) + ca(u, v) + \frac{c^2}{2}a(v, v) - \ell(u) - c\ell(v) \geq F(u) = g(0),$$

where g is a quadratic function of c. Taking derivative of g and set it to zero, we obtain,

$$g'(0) = a(u, v) - \ell(v) = 0,$$

and the proof is complete. ■

1.8.2 Back to the variational formulation of PDEs

Proposition 1.8.2 (2.14) *Let Ω be a bounded open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$. Assume that $c \geq 0$, then the variational problem in (2.11),*

$$\int_\Omega (\nabla u \cdot \nabla v + cuv)\, dx = \int_\Omega f v\, dx, \quad \forall v \in H_0^1(\Omega), \tag{2.35}$$

has a unique solution $u \in H_0^1(\Omega)$. In fact, $|u|_{H^1} \leq C\|f\|_{L^2}$ for some $C > 0$.

Proof. H_0^1 is a Hilbert space. We also have defined the bilinear and linear forms, a and ℓ, respectively, in (2.12). To prove the assertion, we only need to check the hypotheses in Lax-Milgram theory.

To show continuity, notice that, for all $u, v \in H_0^1(\Omega)$,

$$\begin{aligned}|a(u,v)| &\leq \int_\Omega |\nabla u \cdot \nabla v|\, dx + \int_\Omega |cuv|\, dx \\ &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|c\|_{L^\infty} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq C|u|_{H^1(\Omega)} |v|_{H^1(\Omega)},\end{aligned}$$

where we have used the Poincaré's inequality to achieve the last line above.

Next, we show that a is coercive in H_0^1. For all $v \in H_0^1(\Omega)$,

$$a(v,v) = \int_\Omega \left(\|\nabla v\|^2 + cv^2\right) dx \geq \int_\Omega \|\nabla v\|^2 dx = |v|_{H^1(\Omega)}.$$

Finally, we need to check the continuity of the linear functional ℓ. For all $v \in H_0^1(\Omega)$,

$$|\ell(v)| \leq \int_\Omega fv\, dx \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} |v|_{H^1(\Omega)}, \tag{2.37}$$

where we have used the Poincaré's inequality. Since all Lax-Milgram hypotheses are satisfied, there exists a unique solution to (2.35).

From (2.32) with $H = H_0^1$,

$$\alpha \|u\|_H^2 \leq a(u,u) = \ell(u) \leq \|\ell\|_{H'} \|u\|_H$$

we have,

$$|u|_{H^1} \leq \frac{1}{\alpha} \|\ell\|_{H'} \leq \frac{C}{\alpha} \|f\|_{L^2},$$

where the second inequality used (2.37). ■

Remark 1.8.3 (2.15) *Here are several remarks:*

i Since a is symmetric, the weak solution can also be obtained by solving the minimization problem in (2.33).

ii The proof in Prop. 2.14 essentially gives the same result for the non-homogeneous Dirichlet problem as well.

iii One can also show that there exists a unique solution $u \in H^1(\Omega)$ for the Neumann problem in (2.20) with additional conditions, $c > 0$ and $g \in L^2(\partial\Omega)$ in addition to the conditions for the Proposition 2.14.

Remark 1.8.4 (2.17) ***Regularity of the solutions.*** *For $1D$ problem, it is immediate from Prop. 2.1 that $u \in H^2(\Omega)$. For example, for the homogeneous Dirichlet problem, one can see that,*

$$\begin{aligned}\|u''\|_{L^2} &\leq \|cu\|_{L^2} + \|f\|_{L^2} \\ &\leq \|c\|_{L^\infty} \|u\|_{L^2} + \|f\|_{L^2} \\ &\leq C\|c\|_{L^\infty} |u|_{H^1} + \|f\|_{L^2} \\ &\leq C\|f\|_{L^2},\end{aligned}$$

where we have used the continuity of the solution in (2.38) in the last inequality above.

Similar type of result holds for higher dimensions and a bounded domain with smooth boundary, and such a result is nontrivial.

Theorem 1.8.5 (2.18) *Let $\Omega \subset \mathbb{R}^d$ be a convex domain, then for any $f \in L^2(\Omega)$, the solution of the variational problem in (2.13) satisfies,*

$$\|u\|_{H^2} \le C\|f\|_{L^2}, \tag{2.39}$$

for some constant $C > 0$. In fact, if Ω has smooth boundary, one can generalize the regularity as follows. For any $f \in H^k$, the solution to the variational problem $u \in H^{k+2} \cap H_0^1$, and

$$\|u\|_{H^{k+2}} \le C\|f\|_{H^k}. \tag{2.40}$$

1.9 2.8 Estimates for General Finite Element Approximation

Let u be the solution to the variational problem (2.6.2) and u_h be the solution to the approximation problem (2.6.3). We now want to estimate the error $\|u - u_h\|_V$. We do so by the following theorem.

Theorem 1.9.1 ((2.8.1) (Céa)) *Suppose the conditions (2.6.1) hold and that u solves (2.6.2). For the finite element variational problem (2.6.3) we have*

$$\|u - u_h\|_V \le \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V, \tag{2.8.2}$$

where C is the continuity constant and α is the coercivity constant of $a(\cdot,\cdot)$ on V.

Proof. Since $a(u, v) = F(v)$ for all $v \in V$ and $a(u_h, v) = F(v)$ for all $v \in V_h$ we have (by subtracting)

$$a(u - u_h, v) = 0 \quad \forall v \in V_h. \tag{2.8.3}$$

For all $v \in V_h$,

$$\begin{aligned}
\alpha\|u - u_h\|_V^2 &\le a(u - u_h, u - u_h) \quad \text{(by coercivity)} \\
&= a(u - u_h, u - v) + a(u - u_h, v - u_h) \\
&= a(u - u_h, u - v) \quad \text{(since } v - u_h \in V_h\text{)} \\
&\le C\|u - u_h\|_V \|u - v\|_V. \quad \text{(by continuity)}
\end{aligned}$$

Hence,

$$\|u - u_h\|_V \le \frac{C}{\alpha}\|u - v\|_V \quad \forall v \in V_h. \tag{2.8.4}$$

Therefore,

$$\begin{aligned}
\|u - u_h\|_V &\le \frac{C}{\alpha} \inf_{v \in V_h} \|u - v\|_V \\
&= \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V. \quad \textbf{(since } V_h \textbf{ is closed)}
\end{aligned}$$

■

Remark 1.9.2 ((2.8.5))

1. *Céa' s Theorem shows that u_h is quasi-optimal in the sense that the error $\|u-u_h\|_V$ is proportional to the best it can be using the subspace V_h.*

2. ***(重要和前面有联系)*** *In the symmetric case, we proved*

$$\|u-u_h\|_E = \min_{v\in V_h} \|u-v\|_E.$$

Hence,

$$\begin{aligned}
\|u-u_h\|_V &\leq \frac{1}{\sqrt{\alpha}}\|u-u_h\|_E \quad \textbf{\textit{(coercivity)}} \\
&= \frac{1}{\sqrt{\alpha}} \min_{v\in V_h} \|u-v\|_E \\
&\leq \sqrt{\frac{C}{\alpha}} \min_{v\in V_h} \|u-v\|_V \quad \textbf{\textit{(continuity of }} a(u,v)\textit{)} \\
&\leq \frac{C}{\alpha} \min_{v\in V_h} \|u-v\|_V,
\end{aligned}$$

the result of Céa's Theorem. This is really the remark about the relationship between the two formulations, namely, that one can be derived from the other.

Definition 1.9.3 (3.1 (Conforming approximation)) *Let V be a Hilbert space and $\{V_h \subset V\}$ for some parameter h (associated with discretization) be a sequence of finite-dimensional vector space. This sequence of subspaces is conforming if for all $u \in V$, there exists a sequence $\{v_h \in V_h\}$ such that $\|u-v_h\|_V \to 0$, as $h \to 0$.*

Proposition 1.9.4 *If V_h is a conforming approximation sequence, each component is finite-dimensional and hence closed subspace of V, then $u_h \to u$ in V with a priori estimate,*

$$\|u-u_h\|_V \leq \frac{M}{\alpha} \inf_{\nu_h\in V_h} \|u-\nu_h\|_V \leq \frac{M}{\alpha}\|u-\nu_h'\|_V \to 0, \quad \textit{as } h \to 0, \tag{1.1}$$

for some $\nu_h' \in V_h$, by the definition of conforming approximation.

Next, we provide **the priori and posterior error estimates** from Harlim's note. We need some notations here. Since V_h is closed, it is a Hilbert space defined with respect to the scalar product in V. The Lax-Milgram hypotheses are also satisfied in V_h, so there exists a unique solution $u_h \in V_h$, such that

$$a(u_h, w_h) = \ell(w_h), \quad \forall w_h \in V_h. \tag{3.39}$$

But since $w_h \in V$,

$$a(u, w_h) = \ell(w_h), \tag{3.40}$$

where $u \in V$. Subtracting the two variational equations, we achieve,

$$a(u-u_h, w_h) = 0, \quad \forall w_h \in V_h. \tag{3.41}$$

Theorem 1.9.5 (4.7 (A priori error estimate)) *Let u and u_h satisfy (3.40) and (3.39), respectively. Then*

$$|u-u_h|_{H^1(\Omega)} \leq Ch\|u''\|_{L^2(\Omega)}. \tag{4.47}$$

Furthermore, we can also deduce that,

$$\|u-u_h\|_{L^2(\Omega)} \leq Ch^2\|u''\|_{L^2(\Omega)}. \tag{4.48}$$

Proof. Since u is a variational solution of (3.40), by Theorem 2.1 for one-dimension, it is clear that $u \in H^2(\Omega)$. The bound in (4.47) can be directly obtained by Céa's Lemma in 3.3 and the interpolation error bound in (4.45),

$$|u - u_h|_{H^1} \leq \frac{M}{\alpha} \inf_{\nu_h \in V_h} |u - \nu_h|_{H^1} \leq \frac{M}{\alpha} h \|u''\|_{L^2},$$

where we take $\nu_h = \Pi_h u$. To derive the second error bound (4.48), we use the **Aubin-Nitsche trick**. Consider the variational problem corresponding to an auxiliary problem,

$$-\phi'' + c\phi = e = u_h - u, \tag{4.49}$$

which is to find $\phi \in H_0^1$ such that $a(\phi, w) = \langle e, w \rangle_{L^2}, \forall w \in H_0^1$. From the regularity theory (Theorem 2.18), $\|\phi''\|_{L^2} \leq C\|e\|_{L^2}$. Take $w = e$,

$$\begin{aligned}
\langle e, e \rangle_{L^2} &= a(\phi, e) \\
&= a(\phi - \Pi_h \phi, e) \\
&\leq M \|e'\|_{L^2} \|(\phi - \Pi_h \phi)'\|_{L^2},
\end{aligned}$$

where in the second inequality, we have used the fact that $a(v, e) = a(e, v) = 0$ for any $v \in V_h$ (since a is symmetric in this case), and the last inequality is due to the continuity of the bilinear form a with respect to $|\cdot|_{H^1}$ norm in H_0^1. Using (4.45) and the regularity of the solution,

$$\|e\|_{L^2}^2 \leq CMh \|e'\|_{L^2} \|\phi''\|_{L^2(\Omega)} \leq C'h \|e'\|_{L^2} \|e\|_{L^2},$$

for some $C' > 0$, and (4.47), we obtain,

$$\|e\|_{L^2} \leq C'h \|e'\|_{L^2} \leq C''h^2 \|u''\|_{L^2},$$

which completes the proof. ■

Theorem 1.9.6 (4.8 (A posteriori error estimate)) *Let u and u_h satisfy (3.40) and (3.39), respectively. Then*

$$\|u - u_h\|_{L^2(\Omega)} \leq C \sum_{i=0}^{N} h_i^2 \|R_h\|_{L^2((x_i, x_{i+1}))}, \tag{4.50}$$

for some constant $C > 0$, with finite element residual, $R_h := -u_h'' + cu_h - f$.

Proof. Let ϕ be a solution of the auxiliary problem in (4.49). Following the same deduction, let $e = u_h - u$ we can write,

$$\begin{aligned}
\langle e, e \rangle_{L^2} &= a(e, \phi) \\
&= a(e, \phi - \Pi_h \phi) \\
&= a(u_h, \phi - \Pi_h \phi) - a(u, \phi - \Pi_h \phi) \\
&= a(u_h, \phi - \Pi_h \phi) - \langle f, \phi - \Pi_h \phi \rangle_{L^2} \\
&= \int_a^b (-u_h'' + cu_h - f)(\phi - \Pi_h \phi)\, dx \\
&= \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} (-u_h'' + cu_h - f)(\phi - \Pi_h \phi)\, dx.
\end{aligned}$$

By Cauchy-Schwartz, the interpolation error in (4.46), and the regularity $\|\phi''\|_{L^2} \leq C\|u_h - u\|_{L^2}$, we obtain

$$\begin{aligned}
\|u_h - u\|^2_{L^2((a,b))} &\leq \sum_{i=0}^{N} \|R_h\|_{L^2((x_i,x_{i+1}))} \|\phi - \Pi_h\phi\|_{L^2((x_i,x_{i+1}))} \\
&\leq \sum_{i=0}^{N} \|R_h\|_{L^2((x_i,x_{i+1}))} h_i^2 \|\phi''\|_{L^2((a,b))} \\
&\leq C \sum_{i=0}^{N} \|R_h\|_{L^2((x_i,x_{i+1}))} h_i^2 \|u_h - u\|_{L^2((a,b))},
\end{aligned}$$

and the proof is completed. ■