MATH2103: Lecture Note on Numerical Solution of Partial Differential Equations

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Chapter 1

Sobolev Spaces

This chapter is devoted to developing function spaces that are used in the variational formulation of differential equations. We begin with a review of Lebesgue integration theory, upon which our notion of "variational" or "weak" derivative rests. Functions with such "generalized" derivatives make up the spaces commonly referred to as Sobolev spaces. We develop only a small fraction of the known theory for these spaces—just enough to establish a foundation for the finite element method.

1.1 Review of Lebesgue Integration Theory

We will now review the basic concepts of Lebesgue integration theory, cf. (Halmos 1991), (Royden 1988) or (Rudin 1987). By "domain" we mean a Lebesgue-measurable (usually either open or closed) subset of $\mathbb{R}^n$ with non-empty interior. We restrict our attention for simplicity to real-valued functions, f, on a given domain, Ω, that are Lebesgue measurable; by

$$\int_\Omega f(x)dx$$

we denote the Lebesgue integral of f (dx denotes Lebesgue measure). For $1 \le p < \infty$, let

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx\right)^{1/p},$$

and for the case $p = \infty$ set

$$\|f\|_{L^\infty(\Omega)} := \text{ess sup}\left\{|f(x)| : x \in \Omega\right\}.$$

In either case, we define the **Lebesgue spaces**

$$L^p(\Omega) := \{f : \|f\|_{L^p(\Omega)} < \infty\}. \tag{1.1.1}$$

To avoid trivial differences between functions, we identify two functions, f and g, that satisfy $\|f-g\|_{L^p(\Omega)} = 0$. For example, take $n = 1$, $\Omega = [-1, 1]$ and

$$f(x) := \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad g(x) := \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} \tag{1.1.2}$$

Since f and g differ only on a set of measure zero (one point, in this case), we view them as representing the same function. With a small ambiguity of notation, we then think of $L^p(\Omega)$ as a set of **equivalence classes** 等价类of functions with respect to this identification. There are some famous (and useful) inequalities that hold for the functionals defined above:

Minkowski's Inequality For $1 \le p \le \infty$ and $f, g \in L^p(\Omega)$, we have

$$\|f+g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{1.1.3}$$

Hölder's Inequality For $1 \le p, q \le \infty$ such that $1 = 1/p + 1/q$, if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}. \tag{1.1.4}$$

Schwarz' Inequality This is simply Hölder's inequality in the special case $p = q = 2$, viz. if $f, g \in L^2(\Omega)$ then $fg \in L^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|dx \le \|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}. \tag{1.1.5}$$

In view of Minkowski's inequality and the definitions of $\|\cdot\|_{L^p(\Omega)}$, the space $L^p(\Omega)$ is closed under linear combinations, i.e., it is a linear (or vector) space. Moreover, the functionals $\|\cdot\|_{L^p(\Omega)}$ have properties that classify them as norms.

Definition 1.1.1 ***(1.1.6)*** *Given a linear (vector) space V, a norm, $\|\cdot\|$, is a function on V with values in the non-negative reals having the following properties:*

$$\begin{aligned}
&i) && \|v\| \ge 0 \quad \forall v \in V \\
& && \|v\| = 0 \iff v = 0 \\
&ii) && \|c \cdot v\| = |c| \cdot \|v\| \quad \forall c \in \mathbb{R}, v \in V, \quad \textit{and} \\
&iii) && \|v+w\| \le \|v\| + \|w\| \quad \forall v, w \in V \quad (\textit{the triangle inequality}).
\end{aligned}$$

A norm, $\|\cdot\|$, can be used to define a notion of distance, or metric, $d(v, w) = \|v - w\|$ for points $v, w \in V$. A vector space endowed with the topology induced by this metric is called a **normed linear space**. Recall that a metric space, V, is called **complete** if every **Cauchy sequence** $\{v_j\}$ of elements of V has a limit $v \in V$. For a normed linear space, a Cauchy sequence is one such that $\|v_j - v_k\| \to 0$ as $j, k \to \infty$, and completeness means that $\|v - v_j\| \to 0$ as $j \to \infty$. The following definition encapsulates some key features of linear spaces of infinite dimensions needed for theoretical development.

Definition 1.1.2 ***(1.1.7)*** *A normed linear space $(V, \|\cdot\|)$ is called a **Banach space** if it is complete with respect to the metric induced by the norm, $\|\cdot\|$.*

Theorem 1.1.3 ***(1.1.8)*** *For $1 \le p \le \infty$, $L^p(\Omega)$ is a Banach space.*

This theorem (whose proof may be found in the references at the beginning of this section) is a cornerstone of Lebesgue integration theory. Note that it incorporates both Minkowski's inequality and a limit theorem for the Lebesgue integral; that is, if $f_j \to f$ in $L^p(\Omega)$ then (cf. exercise 1.x.3)

$$\int_\Omega |f_j(x)|^p dx \to \int_\Omega |f(x)|^p dx \quad \text{as} \quad j \to \infty. \tag{1.1.9}$$

However, Hölder's inequality and more subtle limit theorems are not reflected in this characterization of the Lebesgue spaces.

A key reason that the Lebesgue integral is preferred over the Riemann integral is the aspect of "completeness" that it enjoys, i.e., **that appropriate limits of integrable functions are integrable, a property that the Riemann integral does not have**.

Example 1.1.4 *For example, we can easily evaluate an improper integral to determine that the function* $\log x$ *has a finite integral on any finite interval of the form* $[0, a]$. *Correspondingly, if* $\{r_n : n = 1, 2, \ldots\}$ *is dense in the interval* $[0, 1]$, *then the functions*

$$f_j(x) := \sum_{n=1}^{j} 2^{-n} \log |x - r_n| \tag{1.1.10}$$

all have improper integrals, and one easily sees that

$$\left| \int_0^1 f_j(x)dx \right| \leq 2 \int_0^1 \log |x - r_n| dx. \tag{1.1.11}$$

Therefore, the "limit" function

$$f(x) := \sum_{n=1}^{\infty} 2^{-n} \log |x - r_n| \tag{1.1.12}$$

should have a finite "integral" on $[0, 1]$, *again satisfying*

$$\left| \int_0^1 f(x)dx \right| \leq 2 \int_0^1 |\log x| dx. \tag{1.1.13}$$

Lebesque控制收敛定理 *However,* f *is infinite at some point in any open sub-interval of* $[0, 1]$ *and so it is not Riemann integrable on any sub-interval of* $[0, 1]$. *Thus, it is not possible, even via "improper" Riemann integrals, to determine if* f *has a finite integral. On the other hand, one can show (see exercise 1.x.6) that it is Lebesgue integrable and that (1.1.13) holds.*

1.2 Generalized (Weak) Derivatives

There are several definitions of derivative that are useful in different situations. The "calculus" definition, viz.

$$u'(x) = \lim_{h \to 0} \frac{u(x + h) - u(x)}{h},$$

is a "local" definition, involving information about the function u only near the point x. The variational formulation developed in Chapter 0 takes a more global view, because pointwise values of derivatives are not needed; only derivatives that can be interpreted as functions in the Lebesgue space $L^2(\Omega)$ occur. In the previous section, we have seen that pointwise values of functions in Lebesgue spaces are irrelevant (cf. (1.1.2)); a function in one of these spaces is determined only by its global behavior. Thus, **it is natural to develop a global notion of *derivative* more suited to the Lebesgue spaces. We do so using a "duality" technique, defining derivatives for a class of not-so-smooth functions** (see Definition 1.2.3) by comparing them with very-very-smooth functions (introduced in Definition 1.2.1).

First, let us introduce some short-hand notation for (calculus) partial derivatives, the **multi-index** notation. A multi-index, α, is an n-tuple of non-negative integers, α_i. The length of α is given by

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

For $\phi \in C^\infty$, denote by

$$D^\alpha\phi, \quad D_x^\alpha\phi, \quad \left(\frac{\partial}{\partial x}\right)^\alpha \phi, \quad \phi^{(\alpha)}, \quad \text{and} \quad \partial_x^\alpha\phi$$

the usual (pointwise) partial derivative

$$\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \phi.$$

Given a vector $(x_1, \ldots, x_n)$, we define $x^\alpha := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Note that if x is replaced formally by the symbol $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, then this definition of x^α is consistent with the previous definition of $\left(\frac{\partial}{\partial x}\right)^\alpha$. Note that the *order* of this derivative is given by $|\alpha|$.

Next, let us introduce the concept of the **support** of a function defined on some domain in $\mathbb{R}^n$. For a continuous function, u, this is the closure of the (open) set $\{x : u(x) \neq 0\}$. If this is a compact set (i.e., if it is bounded) and it is a subset of the **interior** of a set, Ω, then u is said to have "compact support" with respect to Ω. (Outside the support of a function, it is natural to define it to be zero, thus extending it to be defined on all of $\mathbb{R}^n$.) When Ω is a bounded set, it is equivalent to say that u vanishes in a neighborhood of $\partial\Omega$.

Definition 1.2.1 ((1.2.1)) *Let Ω be a domain in $\mathbb{R}^n$. Denote by $D(\Omega)$ or $C_0^\infty(\Omega)$ the set of $C^\infty(\Omega)$ functions with compact support in Ω.*

Before proceeding any further, it would be wise to verify that we have not just introduced a vacuous definition, which we do in the following:

Example 1.2.2 ((1.2.2)) *Define*

$$\phi(x) := \begin{cases} e^{1/(|x|^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}.$$

We claim that, for any multi-index α, $\phi^{(\alpha)}(x) = P_\alpha(x)\phi(x)/(1-|x|^2)^{|\alpha|}$ for some polynomial P_α, as we now show. For $|x| < 1$, we can differentiate and determine, inductively in α, that $\phi^{(\alpha)}(x) = P_\alpha(x)e^{-t}t^{|\alpha|}$ for some polynomial P_α, where $t = 1/(1-|x|^2)$. Further, $\phi^{(\alpha)}(x) = 0$ for $|x| > 1$. Thus, the formula above for $\phi^{(\alpha)}$ is verified in the case $|x| \neq 1$. Since the exponential increases faster than any finite power, $\phi^{(\alpha)}(x)/(1-|x|^2)^k = P_\alpha(x)t^{|\alpha|+k}/e^t \to 0$ as $|x| \to 1$ (i.e., as $t \to \infty$) for any integer k. Applying (inductively) these facts with $k = 0$ shows that $\phi^{(\alpha)}$ is continuous at $|x| = 1$, and using $k = 1$ shows it is also differentiable there, and has derivative zero. Thus, the claimed formula holds for all x. Moreover, we also see from the argument that $\phi^{(\alpha)}$ is bounded and continuous for all α. Thus, $\phi \in D(\Omega)$ for any open set Ω containing the closed unit ball. By scaling variables appropriately, we see that $D(\Omega) \neq \emptyset$ for any Ω with non-empty interior.

We now use the space D to extend the notion of pointwise derivative to a class of functions larger than C^∞. For simplicity, we restrict our notion of derivatives to the following space of functions (see (Schwartz 1957) for a more general definition).

Definition 1.2.3 ((1.2.3)) *Given a domain Ω, the set of* ***locally integrable*** *functions is denoted by*

$$L^1_{loc}(\Omega) := \{f : f \in L^1(K) \quad \forall \text{ compact } K \subset \text{ interior } \Omega\}.$$

Functions in $L^1_{loc}(\Omega)$ can behave arbitrarily badly near the boundary, e.g., the function $e^{e^{1/\mathbf{dist}(x,\partial\Omega)}} \in L^1_{loc}(\Omega)$**,** although this aspect is somewhat tangential to our use of the space. One notational convenience is that $L^1_{loc}(\Omega)$ **contains all of** $C^0(\Omega)$, without growth restrictions. Finally, we come to our new definition of derivative.

We have $L^1 \subset L^1_{loc}$, $C^0 \subset L^1_{loc}$包含连续函数.

Definition 1.2.4 ((1.2.4)) *We say that a given function $f \in L^1_{loc}(\Omega)$ has a* ***weak derivative****, $D^\alpha_w f$, provided there exists a function $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)\phi^{(\alpha)}(x)dx \quad \forall \phi \in D(\Omega).$$

If such a g exists, we define $D^\alpha_w f = g$.

Example 1.2.5 ((1.2.5)) *Take $n = 1$, $\Omega = [-1, 1]$, and $f(x) = 1 - |x|$. We claim that $D^1_w f$ exists and is given by*

$$g(x) := \begin{cases} 1 & x < 0 \\ -1 & x > 0. \end{cases}$$

To see this, we break the interval $[-1, 1]$ into the two parts in which f is smooth, and we integrate by parts. Let $\phi \in D(\Omega)$. Then

$$\begin{aligned} \int_{-1}^1 f(x)\phi'(x)dx &= \int_{-1}^0 f(x)\phi'(x)dx + \int_0^1 f(x)\phi'(x)dx \\ &= -\int_{-1}^0 (+1)\phi(x)dx + f\phi|_{-1}^0 - \int_0^1 (-1)\phi(x)dx + f\phi|_0^1 \\ &= -\int_{-1}^1 g(x)\phi(x)dx + (f\phi)(0-) - (f\phi)(0+) \quad f\text{在0处连续} \\ &= -\int_{-1}^1 g(x)\phi(x)dx \end{aligned}$$

because f is continuous at 0. One may check (cf. exercise 1.x.10) that $D^j_w f$ ***does not exist for*** *$j > 1$.*

One can see that, roughly speaking, the new definition of derivative is the same as the old one wherever the function being differentiated is regular enough. In particular, **continuity of f in the example was enough to insure existence of a first-order weak derivative, but not second-order.**

This phenomenon depends on the dimension n as well, precluding a simple characterization of the relation between the calculus and weak derivatives, as the following example shows.

Example 1.2.6 ((1.2.6)) *Let ρ be a smooth function defined for $0 < r \leq 1$ satisfying*

$$\int_0^1 |\rho'(r)|r^{n-1}dr < \infty.$$

Define f on $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ via $f(x) = \rho(|x|)$. Then $D^\alpha_x f$ exists for all $|\alpha| = 1$ and is given by

$$g(x) = \rho'(|x|)x^\alpha/|x|.$$

The verification of this is left to the reader in exercise 1.x.12.

This example shows that **the relationship between the calculus and weak derivatives depends on dimension**. That is, whether a function such as $|x|^r$ has a weak derivative depends not only on r but also on n (cf. exercise 1.x.13). However, the following fact (whose proof is left as an exercise) shows that the latter is a generalization of the former.

Proposition 1.2.7 ((1.2.7)) *Let α be arbitrary and let $\psi \in C^{|\alpha|}(\Omega)$. Then the weak derivative $D_w^\alpha \psi$ exists and is given by $D^\alpha \psi$.*

As a consequence of this proposition, we ignore the differences in definition of D and D_w from now on. That is, differentiation symbols will refer to weak derivatives in general, but we will also use classical properties of derivatives of smooth functions as appropriate.

1.3 Sobolev Norms and Associated Spaces

Using the notion of weak derivative, we can generalize the Lebesgue norms and spaces to include derivatives.

Definition 1.3.1 ((1.3.1)) *Let k be a non-negative integer, and let $f \in L^1_{loc}(\Omega)$. Suppose that the weak derivatives $D^\alpha_\omega f$ exist for all $|\alpha| \le k$. Define the **Sobolev norm***

$$\|f\|_{W^k_p(\Omega)} := \left(\sum_{|\alpha| \le k} \|D^\alpha_\omega f\|^p_{L^p(\Omega)} \right)^{1/p}$$

in the case $1 \le p < \infty$, and in the case $p = \infty$

$$\|f\|_{W^k_\infty(\Omega)} := \max_{|\alpha| \le k} \|D^\alpha_\omega f\|_{L^\infty(\Omega)}.$$

*In either case, we define the **Sobolev spaces** via*

$$W^k_p(\Omega) := \left\{ f \in L^1_{loc}(\Omega) : \|f\|_{W^k_p(\Omega)} < \infty \right\}.$$

Example 1.3.2 ***The Sobolev spaces can be related in special cases to other spaces.*** *For example, recall the **Lipschitz norm**, Hölder space with Holder index $\alpha = 1$,*

$$\|f\|_{Lip(\Omega)} = \|f\|_{L^\infty(\Omega)} + \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in \Omega; x \ne y \right\},$$

and the corresponding space of Lipschitz functions

$$Lip(\Omega) = \left\{ f \in L^\infty(\Omega) : \|f\|_{Lip(\Omega)} < \infty \right\}.$$

Then for all dimensions n, we have $Lip(\Omega) = W^1_\infty(\Omega)$ with equivalent norms, at least under certain conditions on the domain Ω (cf. exercises 1.x.15 and 1.x.14). (Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a linear space V are said to be equivalent provided there is a positive constant $C < \infty$ such that $\|v\|_1/C \le \|v\|_2 \le C\|v\|_1 \quad \forall v \in V$.) Moreover, for $k > 1$

$$W^k_\infty(\Omega) = \left\{ f \in C^{k-1}(\Omega) : f^{(\alpha)} \in Lip(\Omega), \forall |\alpha| \le k - 1 \right\}.$$

Example 1.3.3 *In one dimension ($n = 1$), the space $W_1^1(\Omega)$ can be characterized as the set of absolutely continuous functions on an interval Ω (cf. (Hartman & Mikusinski 1961) and exercises 1.x.17 and 1.x.22).*

It is easy to see that $\|\cdot\|_{W_p^k(\Omega)}$ is a norm. Thus, $W_p^k(\Omega)$ is by definition a normed linear space. The following theorem shows that it is complete.

Theorem 1.3.4 ((1.3.2)) *The Sobolev space $W_p^k(\Omega)$ is a Banach space.*

Proof. Let $\{v_j\}$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{W_p^k(\Omega)}$. Since the $\|\cdot\|_{W_p^k(\Omega)}$ norm is just a combination of $\|\cdot\|_{L^p(\Omega)}$ norms of weak derivatives, it follows that, for all $|\alpha| \leq k$, $\{D_w^\alpha v_j\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{L^p(\Omega)}$. Thus, Theorem 1.1.8 implies the existence of $v^\alpha \in L^p(\Omega)$ such that $\|D_w^\alpha v_j - v^\alpha\|_{L^p(\Omega)} \to 0$ as $j \to \infty$. In particular, $v_j \to v^{(0,\ldots,0)} =: v$ in $L^p(\Omega)$. What remains to check is that $D_w^\alpha v$ exists and is equal to v^α.

First, note that if $w_j \to w$ in $L^p(\Omega)$, then for all $\phi \in D(\Omega)$

$$\int_\Omega w_j(x)\phi(x)dx \to \int_\Omega w(x)\phi(x)dx. \tag{1.3.3}$$

This follows from (1.1.9) and Hölder's inequality:

$$\|w_j\phi - w\phi\|_{L^p(\Omega)} \leq \|w_j - w\|_{L^p(\Omega)}\|\phi\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad j \to \infty.$$

Second, to show that $D_w^\alpha v = v^\alpha$, we must show that

$$\int_\Omega v^\alpha \phi dx = (-1)^{|\alpha|}\int_\Omega v\phi^{(\alpha)}dx \quad \forall \phi \in D(\Omega).$$

This follows from the definition of the weak derivative, $D_w^\alpha v_j$, and two applications of (1.3.3):

$$\begin{aligned}\int_\Omega v^\alpha \phi dx &= \lim_{j\to\infty}\int_\Omega (D_w^\alpha v_j)\phi dx \\ &= \lim_{j\to\infty}(-1)^{|\alpha|}\int_\Omega v_j\phi^{(\alpha)}dx = (-1)^{|\alpha|}\int_\Omega v\phi^{(\alpha)}dx,\end{aligned}$$

where we have used (1.3.3) in the first and last equality. ■

There is another potential definition of Sobolev space that could be made. Let $H_p^k(\Omega)$ denote the closure of $C^k(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{W_p^k(\Omega)}$. In the case $p = \infty$, we have $H_\infty^k = C^k$, and this is not the same as $W_\infty^k(\Omega)$. Indeed, we have already identified the latter as being related to certain Lipschitz spaces. However, for $1 \leq p < \infty$, it turns out that $H_p^k(\Omega) = W_p^k(\Omega)$. The following result was proved in a paper (Meyers & Serrin 1964) that is celebrated both for the importance of the result and the brevity of its title.

Theorem 1.3.5 ((1.3.4)) *Let Ω be any open set. Then $C^\infty(\Omega) \cap W_p^k(\Omega)$ is dense in $W_p^k(\Omega)$ for $p < \infty$.*

For technical reasons it is useful to introduce the following notation for the Sobolev **semi-norms**.

Definition 1.3.6 ((1.3.7)) *For k a non-negative integer and $f \in W_p^k(\Omega)$, let*

$$|f|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k}\|D_\omega^\alpha f\|_{L^p(\Omega)}^p\right)^{1/p}$$

in the case $1 \leq p < \infty$, and in the case $p = \infty$

$$|f|_{W_\infty^k(\Omega)} = \max_{|\alpha|=k}\|D_\omega^\alpha f\|_{L^\infty(\Omega)}.$$

补充Hölder space from Evans，Lipschitz空间是Hölder在$\alpha = 1$下的特例

1.4 Hölder spaces

Before turning to Sobolev spaces, we first discuss the simpler **Hölder spaces**.

Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. We have previously considered the class of Lipschitz continuous functions $u : U \to \mathbb{R}$, which by definition satisfy the estimate

$$|u(x) - u(y)| \leq C|x - y| \quad (x, y \in U) \tag{1}$$

for some constant C. Now (1) of course implies u is continuous, and more importantly provides a uniform modulus of continuity. It turns out to be useful to consider also functions u satisfying a variant of (1), namely

$$|u(x) - u(y)| \leq C|x - y|^\gamma \quad (x, y \in U) \tag{2}$$

for some constant C. Such a function is said to be **Hölder continuous with exponent** γ.

Definition 1.4.1 *(i) If $u : U \to \mathbb{R}$ is* ***bounded and continuous****, we write*

$$\|u\|_{C(U)} := \sup_{x \in U} |u(x)|.$$

(ii) The γ^{th}-Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(U)} := \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\},$$

and the γ^{th}-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}.$$

Definition 1.4.2 *The Hölder space*

$$C^{k,\gamma}(\bar{U})$$

consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})} \tag{1.1}$$

is finite.

So the space $C^{k,\gamma}(\bar{U})$ consists of those functions u that are k-times continuously differentiable and whose k^{th}-partial derivatives are Hölder continuous with exponent γ. Such functions are well-behaved, and furthermore the space $C^{k,\gamma}(\bar{U})$ itself possesses a good mathematical structure:

Theorem 1.4.3 (1) *The space of functions $C^{k,\gamma}(\overline{U})$ is a Banach space.*

The proof is left as an exercise (Problem 1), but let us pause here to make clear what is being asserted. Recall from §D.1 that if X denotes a real linear space, then a mapping $\|\cdot\| : X \to [0, \infty)$ is called a norm provided

(i) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in X$,

(ii) $\|\lambda u\| = |\lambda|\|u\|$ for all $u \in X$, $\lambda \in \mathbb{R}$,

(iii) $\|u\| = 0$ if and only if $u = 0$.

A norm provides us with a notion of convergence: we say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ **converges** to $u \in X$, written $u_k \to u$, if

$$\lim_{k\to\infty} \|u_k - u\| = 0.$$

A **Banach space** is then a normed linear space which is **complete**, that is, within which each Cauchy sequence converges.

So in Theorem 1 we are stating that if we take on the linear space $C^{k,\gamma}(\overline{U})$ the norm $\|\cdot\| = \|\cdot\|_{C^{k,\gamma}(\overline{U})}$, defined by (3), then $\|\cdot\|$ verifies properties (i)-(iii) above, and in addition each Cauchy sequence converges.

补充Hölder space from Evans，Lipschitz空间是Hölder在$\alpha = 1$下的特例。原来的C^k空间已经是完备的了，而Holder space 是为了进一步刻画函数的导数连续性，进而使用Schauder estimate 。

Example 1.4.4 (1.5) *Show that $f(x) = x^{\beta}$ for $\beta \le 1$ on $[0,1]$ is $C^{0,\alpha}$ continuous for $0 < \alpha \le \beta$ but not for $\alpha > \beta$.*

To show that f is $C^{0,\alpha}$ continuous for $0 < \alpha \le \beta$, one only needs to check the Hölder continuity at 0. Away from 0, this function has a derivative so it is definitely Lipschitz, i.e., $C^{0,1}$. By the embedding, it is also $C^{0,\alpha}$ for $\alpha \le 1$. At $x = 0$, we have

$$\frac{|x^{\beta} - 0^{\beta}|}{|x - 0|^{\alpha}} = x^{\beta-\alpha}.$$

When $\alpha \le \beta$, $x^{\beta-\alpha} \to 0$ as $x \to 0^+$, so the Hölder condition holds. On the contrary, if $\alpha > \beta$, $x^{\beta-\alpha} \to \infty$ as $x \to 0^+$, so f is not $C^{0,\alpha}$ continuous.

1.5 Inclusion Relations and Sobolev's Inequality

Given the number of indices defining Sobolev spaces, it is natural to hope that there are inclusion relations to provide some sort of ordering among them. Using the Definition 1.3.1 and exercise 1.x.1, it is easy to derive the following propositions.

Proposition 1.5.1 ((1.4.1)) *Suppose that Ω is any domain, k and m are non-negative integers satisfying $k \le m$, and p is any real number satisfying $1 \le p \le \infty$. Then $W_p^m(\Omega) \subset W_p^k(\Omega)$.*

Proposition 1.5.2 ((1.4.2)) *Suppose that Ω is a bounded domain, k is a non-negative integer, and p and q are real numbers satisfying $1 \le p \le q \le \infty$. Then $W_q^k(\Omega) \subset W_p^k(\Omega)$.*

However, there are more subtle relations among the Sobolev spaces. For example, there are cases when $k < m$ and $p > q$ and $W_q^m(\Omega) \subset W_p^k(\Omega)$. The existence of Sobolev derivatives imply a stronger integrability condition of a function. To set the stage, let us consider an example to give us guidance as to **possible relations among k, m, p, and q for such a result to hold**.

Example 1.5.3 ((1.4.3)) *Let $n \ge 2$, let $\Omega = \{x \in \mathbb{R}^n : |x| < 1/2\}$ and consider the function $f(x) = \log|\log|x||$. From Example 1.2.6 (and exercise 1.x.12), we see that f has first-order weak derivatives*

$$D^{\alpha} f(x) = x^{\alpha}/(|x|^2 \log|x|)$$

($|\alpha| = 1$). From exercise 1.x.5, we see that $D^\alpha f \in L^p(\Omega)$ provided $p \le n$. For example,

$$|D^\alpha f(x)|^n \le \rho(|x|) := 1/(|x|^n |\log |x||^n)$$

satisfies the condition of exercise 1.x.5 because $\rho(r) r^{n-1} = 1/(r|\log r|^n)$ is integrable for all $n \ge 2$ on $[0, 1/2]$. In fact, the change of variables $r = e^{-t}$ gives

$$\int_0^{1/2} \frac{dr}{r|\log r|^n} = \int_{\log 2}^{\infty} \frac{dt}{t^n} < \infty.$$

($n \ge 2$)

Similarly, it is easy to see that $f \in L^p(\Omega)$ for $p < \infty$. Thus, $f \in W_p^1(\Omega)$ for $p \le n$. Note, however, that in no case is $f \in L^\infty(\Omega)$.

This example shows that there are functions that are essentially infinite at points (such points could be chosen as in (1.1.12) to be everywhere dense), yet which have p-th power integrable weak derivatives. Moreover, as the dimension n increases, the integrability power p increases as well. On the other hand, the following result, which will be proved in Chapter 4, shows that if a function has p-th power integrable weak derivatives for sufficiently large p (with n fixed), it must be bounded (and, in fact, can be viewed as being continuous). But before we state the result, we must introduce a regularity condition on the domain boundary for the result to be true.

Definition 1.5.4 ((1.4.4)) *We say Ω has a **Lipschitz boundary** $\partial\Omega$ provided there exists a collection of open sets O_i, a positive parameter ϵ, an integer N and a finite number M, such that for all $x \in \partial\Omega$ the ball of radius ϵ centered at x is contained in some O_i, no more than N of the sets O_i intersect nontrivially, and each domain $O_i \cap \Omega = O_i \cap \Omega_i$ where Ω_i is a domain whose boundary is a graph of a Lipschitz function ϕ_i (i.e., $\Omega_i = \{(x, y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, y < \phi_i(x)\}$) satisfying $\|\phi_i\|_{Lip(\mathbb{R}^{n-1})} \le M$.*

One consequence of this definition is that we can now relate Sobolev spaces on a given domain to those on all of $\mathbb{R}^n$.

下面定理很重要，但看不太懂

Theorem 1.5.5 ((1.4.5)) *Suppose that Ω has a Lipschitz boundary. Then there is an **extension** mapping $E : W_p^k(\Omega) \to W_p^k(\mathbb{R}^n)$ defined for all non-negative integers k and real numbers p in the range $1 \le p \le \infty$ satisfying $Ev|_\Omega = v$ for all $v \in W_p^k(\Omega)$ and*

$$\|Ev\|_{W_p^k(\mathbb{R}^n)} \le C\|v\|_{W_p^k(\Omega)}$$

where C is independent of v.

For a proof of this result, as well as more details concerning other material in this section, see (Stein 1970). Of course, the complementary result is true for any domain, namely, that the natural restriction allows us to view functions in $W_p^k(\mathbb{R}^n)$ as well defined in $W_p^k(\Omega)$. We now return to the question regarding the relationship between Sobolev spaces with different indices.

Theorem 1.5.6 ((1.4.6)) ***(Sobolev's Inequality)*** *Let Ω be an n-dimensional domain with Lipschitz boundary, let k be a positive integer and let p be a real number in the range $1 \le p < \infty$ such that*

$$k \ge n \quad when \quad p = 1$$

$$k > n/p \quad when \quad p > 1.$$

Then there is a constant C such that for all $u \in W_p^k(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{W_p^k(\Omega)}.$$

Moreover, ***there is a continuous function in the $L^\infty(\Omega)$ equivalence class of u.***

This result says that any function with suitably regular weak derivatives may be viewed as **a continuous, bounded function**. Note that Example 1.4.3 shows that the result is sharp, namely, that the condition $k > n/p$ cannot be relaxed (unless $p = 1$), at least when $n \ge 2$. When $n = 1$, Sobolev's inequality says that integrability of first-order derivatives to any power $p \ge 1$ is sufficient to guarantee continuity. The result will be proved as a corollary to our polynomial approximation theory to be developed in Chapter 4. Note that we can apply it to derivatives of functions in Sobolev spaces to derive the following:

Corollary 1.5.7 ((1.4.7)) *Let Ω be an n-dimensional domain with Lipschitz boundary, and let k and m be positive integers satisfying $m < k$ and let p be a real number in the range $1 \le p < \infty$ such that*

$$k - m \ge n \quad when \quad p = 1$$

$$k - m > n/p \quad when \quad p > 1.$$

Then there is a constant C such that for all $u \in W_p^k(\Omega)$

$$\|u\|_{W_\infty^m(\Omega)} \le C\|u\|_{W_p^k(\Omega)}.$$

Moreover, there is a C^m function in the $L^p(\Omega)$ equivalence class of u.

Remark 1.5.8 ((1.4.8)) *If $\partial\Omega$ is not Lipschitz continuous, then neither Theorem 1.4.5 nor Theorem 1.4.6 need hold. For example, let*

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, |y| < x^r\}$$

where $r > 1$, and let $u(x, y) = x^{-\epsilon/p}$, where $0 < \epsilon < r$. Then

$$\sum_{|\alpha|=1} \int_\Omega |D^\alpha u|^p dxdy = c_{\epsilon,p} \int_0^1 x^{-\epsilon-p+r} dx < \infty,$$

provided $p < 1 + r - \epsilon$. In this case, $u \in W_p^1(\Omega)$ but u is not essentially bounded on Ω if $\epsilon > 0$. Choosing ϵ so that it is possible to have $p > 2$, we find that a Lipschitz boundary is necessary for Sobolev's inequality to hold. Since Sobolev's inequality does hold on $\mathbb{R}^n$, it is not possible to extend u to an element of $W_2^1(\mathbb{R}^2)$. Thus, the extension theorem for Sobolev functions can not hold if $\partial\Omega$ is not Lipschitz continuous.

更多有用的Sobolev嵌入定理可以参考整理的doc文件。

1.5.1 Ref. from Harlim's note

以下嵌入定理是常用的在H^m空间中的结论，我们希望解是连续的，因为数值解是连续的。

Theorem 1.5.9 (1.39) *Let Ω be a Lipschitz open subset of $\mathbb{R}^d$. Let $m > d/2$, then $H^m(\Omega) \subset C(\bar{\Omega})$ and there exists a constant $C = C(\Omega) > 0$ such that*

$$\|u\|_{C(\bar{\Omega})} \le C\|u\|_{H^m(\Omega)}, \quad \forall u \in H^m(\Omega).$$

Applying this inequality to distributional derivative of u and $m > d/2 + \ell$, we have

$$\|u\|_{C^\ell(\bar{\Omega})} \le C\|u\|_{H^m(\Omega)}, \quad \forall u \in H^m(\Omega).$$

We also conclude that $H^m(\Omega) \subset C^\ell(\bar{\Omega})$ if $m > d/2 + \ell$.

1.5.2 Ref. from my PDE's note

最深刻的是Sobolev嵌入定理。这个版本从连续空间角度探讨，本质应该和Sussan书上一致。

Let Ω bounded in $\mathbb{R}^n$, $\quad 1 \le p < \infty$. Then

$$W_0^{1,p}(\Omega) \subset \begin{cases} L^{\frac{np}{n-p}}(\Omega), & p < n, \\ C^{0,\alpha}(\Omega), \quad \alpha = 1 - \frac{n}{p}, & p > n, \end{cases}$$

and the embedding is continuous. Moreover, we have

$$W^{j+m,p}(\Omega) \subset \begin{cases} W^{j,q}(\Omega), & p \le q \le \frac{np}{n-mp}, \\ C^{j,\lambda}(\overline{\Omega}), & 0 \le \lambda \le m - \frac{n}{p}. \end{cases}$$

1.6 Review of Chapter 0

At this point we can tie up many of the loose ends from the previous chapter. We see that **the space V introduced there can now be rigorously defined as**

$$\boxed{V = \{v \in W_2^1(\Omega) : v(0) = 0\}}$$

where $\Omega = [0, 1]$, and that this makes sense because Sobolev's inequality guarantees that pointwise values are well defined for functions in $W_2^1(\Omega)$. (In fact, we are allowed to view $W_2^1(\Omega)$ as a subspace of $C_b(\Omega)$, the Banach space of bounded continuous functions.)

The derivation of the variational formulation (0.1.3) for solution of (0.1.1) can now be made rigorous (cf. exercise 1.x.24). We have stated in Sect. 1.3 (see the related exercises) that functions in $W_1^1(\Omega)$, and a *fortiori* those in $W_2^1(\Omega)$, are absolutely continuous, implying that **the Cantor function is not among them**. However, we saw in Example 1.2.5 that **piecewise linear functions have weak derivatives** that are piecewise constant. Thus, we can assert that the spaces S constructed in Sect. 0.4 satisfy $S \subset W_\infty^1(\Omega)$, and therefore that $\boxed{S \subset V}$.

Since Sobolev's inequality implies that $V \subset C_b(\Omega)$, exercise 0.x.8 shows that w in the duality argument leading to Theorem 0.3.5 is well defined (also see exercise 1.x.23).

In the error estimates for $u - u_S$ we make reference to the $L^2(\Omega)$ norm, $\|\cdot\|$, of second derivatives of functions. We can now make those expressions rigorous by interpreting them in the context of functions in $W_2^2(\Omega)$. In particular, we can re-state the approximation assumption (0.3.4) as

$$\exists \epsilon < \infty \quad \text{such that} \quad \inf_{v \in S} \|w - v\|_E \le \epsilon \|w''\| \quad \forall w \in W_2^2(\Omega), \tag{0.3.4bis}$$

and the only condition needed for Theorem 0.4.5 to hold is $f \in L^2(\Omega)$ (cf. exercise 1.x.23).

Finally, in the proofs in Chapter 0 we argued that $a(v, v) = 0$ implied $v \equiv 0$. While it certainly implies that the weak derivative of v is zero as a function in $L^2(\Omega)$, to conclude that v must be constant requires a notion of **coercivity** that will subsequently be developed in detail. For now, consider the following simple case. From exercise 1.x.16, we know that, for all $v \in V$, we can write

$$v(x) = \int_0^x D_w^1 v(s) ds$$

and use Schwarz' inequality (cf. the proof of Theorem 0.4.5) to estimate

$$|v(x)|^2 \leq x \cdot \int_0^x D_w^1 v(s)^2 ds.$$

Integrating with respect to x yields

$$\|v\|_{L^2(\Omega)}^2 \leq \frac{1}{2} a(v, v).$$

In particular, this shows that if $v \in V$ satisfies $a(v, v) = 0$ then $v = 0$ as an element of $L^2(\Omega)$. Moreover, recalling the definition of the $W_2^1(\Omega)$ norm, we see that

$$\|v\|_{W_2^1(\Omega)}^2 \leq \frac{3}{2} a(v, v) \quad \forall v \in V. \tag{1.5.1}$$

Thus, we can conclude that vanishing of $a(v, v)$ for $v \in V$ implies v is the zero element in V (or $W_2^1(\Omega)$). Inequality (1.5.1) is a **coercivity inequality** for the bilinear form $a(\cdot, \cdot)$ on the space V. Note that this inequality is only valid on the subspace V of $W_2^1(\Omega)$, not all of $W_2^1(\Omega)$, since it fails if we take v to be a non-zero constant function (regardless of what constant would be substituted for $\frac{3}{2}$).

1.6.1 Generalization to non-homogeneous case from Harlim's note

Here, we will consider the following boundary value problem on a Lipschitz, connected, open domain $\Omega \subset \mathbb{R}^d$.

$$\begin{aligned} -\Delta u + cu &= f, \quad \text{in } \Omega \\ u &= g, \quad \text{on } \partial\Omega \end{aligned} \tag{2.10}$$

For simplicity, we let $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$, $g = 0$.

It is easy to check that if $\boxed{u \in H^2(\Omega)}$ solves the PDE in (2.10), then for any $v \in H_0^1(\Omega)$, we have the variational formulation of (2.10),

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_\Omega c(x)u(x)v(x)\, dx = \int_\Omega f(x)v(x)\, dx. \tag{2.11}$$

To see this, use the definition of Sobolev space and Green's formula. What we just see is that for $f \in L^2$, $u \in H^2(\Omega)$ solves the variational problem in (2.11).

Proposition 1.6.1 (2.1) *Let $u \in V$ be a solution of the variational problem in (2.13), then we have,*

$$-\Delta u + cu = f$$

in the sense of $\mathcal{D}'(\Omega)$ and $\Delta u \in L^2(\Omega)$.

Proof. Take $\varphi \in \mathcal{D}(\Omega) \subset H_0^1(\Omega)$ as a test function. Then,

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \langle -\Delta u, \varphi \rangle,$$

using the result in (1.7). Since $cu, f \in L^1_{loc}(\Omega)$, it is clear that they are distributions, so

$$\langle cu, \varphi \rangle = \int_\Omega cu\varphi \, dx \quad \text{and} \quad \langle f, \varphi \rangle = \int_\Omega f\varphi \, dx.$$

By the definition of linear forms in (2.12) and the variational formula in (2.13), it is clear that,

$$\langle -\Delta u + cu - f, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

which means, $-\Delta u + cu - f = 0$ in the sense of $\mathcal{D}'(\Omega)$. Indeed, $\Delta u = cu - f \in L^2(\Omega)$. ■

We can extend this result to non-homogeneous boundary condition.

Proposition 1.6.2 (2.3) *Let $u \in H_0^1(\Omega)$ be a solution of the variational problem,*

$$a(u, v) = \ell(v) - a(\tilde{g}, v), \quad \forall v \in H_0^1(\Omega), \tag{2.17}$$

where $\tilde{g} \in H^1(\Omega)$ such that $\gamma_0(\tilde{g}) = g \in H^{1/2}(\partial\Omega)$. Then, $w = u + \tilde{g}$ solves

$$-\Delta w + cw = f$$

in the sense of $\mathcal{D}'(\Omega)$ and $w = g$ on $\partial\Omega$.

Proof. From the hypothesis,

$$\int_\Omega (\nabla u \cdot \nabla v + cuv) \, dx = \int_\Omega (f - c\tilde{g}) v \, dx - \int_\Omega \nabla \tilde{g} \cdot \nabla v \, dx.$$

Since $u + \tilde{g} \in H^1(\Omega)$, using (I.7),

$$\langle -\Delta(u + \tilde{g}), v \rangle = \int_\Omega \nabla(u + \tilde{g}) \cdot \nabla v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \nabla \tilde{g} \cdot \nabla v \, dx \tag{2.18}$$

for any $v \in \mathcal{D}(\Omega)$. Since $cu, c\tilde{g}, f \in L^1_{\text{loc}}(\Omega)$, they are distributions, and it is clear that,

$$\langle -\Delta(u + \tilde{g}) + c(u + \tilde{g}) - f, v \rangle = 0, \tag{2.19}$$

and $\gamma_0(u) = 0$ since $u \in H_0^1(\Omega)$. Let $w = u + \tilde{g}$, then the proof is completed. ■

1.7 Trace Theorems

In the previous section we saw that it was possible to interpret the "boundary condition" $v(0) = 0$ in the definition of the space V using Sobolev's inequality. As a guide to the higher-dimensional cases of interest later, **this is somewhat misleading, in that Sobolev's inequality, as presented in Sect. 1.4, will not suffice to interpret boundary conditions for higher-dimensional problems.** For example, we have already seen in Example 1.4.3 that, when $n \geq 2$, the analogue of the space V, namely $W_2^1(\Omega)$, contains unbounded functions. **Thus, we cannot interpret the boundary conditions in a pointwise sense, and Sobolev's inequality will have to be augmented in a substantial way to apply when $n \geq 2$.**

On the other hand, the function in Example 1.4.3 can be interpreted as an L^p function on any line in $\mathbb{R}^2$ since the function $\log|\log|\cdot|$ is p-th power integrable in one dimension.

The correct interpretation of the situation is as follows. The boundary $\partial\Omega$ of an n-dimensional domain Ω can be interpreted as **an $n-1$-dimensional object**, a **manifold**. When $n = 1$ it consists of distinct points—the zero-dimensional case of a manifold. Sobolev's inequality gives conditions under which point values are well defined for functions in a Sobolev space, and thus for boundary values in the one-dimensional case. **For higher dimensional problems, we must seek an interpretation of restrictions of Sobolev-class functions to manifolds of dimension $n-1$, and in particular it should make good sense (say, in a Lebesgue class) for functions in $W_2^1(\Omega)$.**

We begin with a simple example to explain the ideas. Let Ω denote the unit disk in $\mathbb{R}^2$:

$$\Omega = \{(x, y) : x^2 + y^2 < 1\} = \{(r, \theta) : 0 \le r < 1, 0 \le \theta < 2\pi\}.$$

Let $u \in C^1(\overline{\Omega})$, and consider its restriction to $\partial\Omega$ as follows:

$$\begin{aligned}
u(1,\theta)^2 &= \int_0^1 \frac{\partial}{\partial r}\left(r^2 u(r,\theta)^2\right) dr \\
&= \int_0^1 2\left(r^2 u u_r + r u^2\right)(r,\theta) dr \\
&= \int_0^1 2\left(r^2 u \nabla u \cdot \frac{(x,y)}{r} + r u^2\right)(r,\theta) dr \\
&\le \int_0^1 2\left(r^2 |u||\nabla u| + r u^2\right)(r,\theta) dr \\
&\le \int_0^1 2\left(|u||\nabla u| + u^2\right)(r,\theta) r dr. \quad (r \le 1)
\end{aligned}$$

Integrating with respect to θ and using polar coordinates (cf. exercise 1.x.4), we find

$$\int_{\partial\Omega} u^2 d\theta \le 2 \int_\Omega \left(|u||\nabla u| + u^2\right) dxdy,$$

where we define the boundary integral (and corresponding norm) in the obvious way:

$$\int_{\partial\Omega} u^2 d\theta := \int_0^{2\pi} u(1,\theta)^2 d\theta =: \|u\|_{L^2(\partial\Omega)}^2. \tag{1.6.1}$$

Using Schwarz' inequality, we have

$$\|u\|_{L^2(\partial\Omega)}^2 \le 2\|u\|_{L^2(\Omega)} \left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + 2\int_\Omega u^2 dxdy.$$

The **arithmetic-geometric mean inequality** (cf. exercise 1.x.32), implies that

$$\left(\int_\Omega |\nabla u|^2 dxdy\right)^{1/2} + \left(\int_\Omega u^2 dxdy\right)^{1/2} \le \left(2\int_\Omega \left(|\nabla u|^2 + u^2\right) dxdy\right)^{1/2}.$$

Therefore,

$$\boxed{\|u\|_{L^2(\partial\Omega)} \le \sqrt[4]{8}\|u\|_{L^2(\Omega)}^{1/2} \|u\|_{W_2^1(\Omega)}^{1/2}.} \tag{1.6.2}$$

This is an inequality analogous to Sobolev's inequality, Theorem 1.4.6, except that the $L^\infty(\Omega)$ norm on the left-hand side of the inequality has been replaced by $\|u\|_{L^2(\partial\Omega)}$. Although we have only proved the inequality for smooth u, we will see that **it makes sense for all** $u \in W_2^1(\Omega)$ (稠密性技巧), and correspondingly that for such u the restriction $u|_{\partial\Omega}$ makes sense as a function in $L^2(\partial\Omega)$. But first, we should say what we mean by the latter space. Using Definition 1.6.1, we can identify it simply with $L^2([0, 2\pi])$ using the coordinate mapping $\theta \to (\cos\theta, \sin\theta)$. (More general boundaries will be discussed shortly.) We can now use inequality (1.6.2) to prove the following result.

Proposition 1.7.1 ((1.6.3)) *Let Ω denote the unit disk in $\mathbb{R}^2$. For all $u \in W_2^1(\Omega)$, the restriction $u|_{\partial\Omega}$ may be interpreted as a function in $L^2(\partial\Omega)$ satisfying (1.6.2).*

Proof.

We will use (1.6.2) three times in the proof, which so far has only been derived for smooth functions. However, in view of Remark 1.3.5, **such functions are dense in** $W_2^1(\Omega)$, so we may pick a sequence $u_j \in C^1(\overline{\Omega})$ such that $\|u - u_j\|_{W_2^1(\Omega)} \leq 1/j$ for all j. By (1.6.2) and the triangle inequality,

$$\begin{aligned}\|u_k - u_j\|_{L^2(\partial\Omega)} &\leq \sqrt[4]{8}\|u_k - u_j\|_{L^2(\Omega)}^{1/2}\|u_k - u_j\|_{W_2^1(\Omega)}^{1/2} \\ &\leq \sqrt[4]{8}\|u_k - u_j\|_{W_2^1(\Omega)} \leq \sqrt[4]{8}\left(\frac{1}{j} + \frac{1}{k}\right)\end{aligned}$$

for all j and k, so that $\{u_j\}$ is a Cauchy sequence in $L^2(\partial\Omega)$. Since this space is complete, there must be a limit $v \in L^2(\partial\Omega)$ such that $\|v - u_j\|_{L^2(\partial\Omega)} \to 0$ as $j \to \infty$. We define

$$u|_{\partial\Omega} := v.$$

The first thing we need to check is that this definition does not depend on the particular sequence that we chose. So suppose that v_j is another sequence of $C^1(\overline{\Omega})$ functions that satisfy $\|u - v_j\|_{W_2^1(\Omega)} \to 0$ as $j \to \infty$. Using the triangle inequality a few times and (1.6.2) again, we see that

$$\begin{aligned}\|v - v_j\|_{L^2(\partial\Omega)} &\leq \|v - u_j\|_{L^2(\partial\Omega)} + \|u_j - v_j\|_{L^2(\partial\Omega)} \\ &\leq \|v - u_j\|_{L^2(\partial\Omega)} + \sqrt[4]{8}\|u_j - v_j\|_{W_2^1(\Omega)} \\ &\leq \|v - u_j\|_{L^2(\partial\Omega)} + \sqrt[4]{8}\left(\|u_j - u\|_{W_2^1(\Omega)} + \|u - v_j\|_{W_2^1(\Omega)}\right) \to 0\end{aligned}$$

as $j \to \infty$. Thus, $u|_{\partial\Omega}$ is well defined in $L^2(\partial\Omega)$. All that remains is to check that (1.6.2) holds for u. Again, we use the validity of it for smooth functions:

$$\begin{aligned}\|u\|_{L^2(\partial\Omega)} &= \|v\|_{L^2(\partial\Omega)} = \lim_{j\to\infty}\|u_j\|_{L^2(\partial\Omega)} \\ &\leq \lim_{j\to\infty}\sqrt[4]{8}\|u_j\|_{L^2(\Omega)}^{1/2}\|u_j\|_{W_2^1(\Omega)}^{1/2} = \sqrt[4]{8}\|u\|_{L^2(\Omega)}^{1/2}\|u\|_{W_2^1(\Omega)}^{1/2}.\end{aligned}$$

This completes the proof of the proposition. Note that (1.6.2) was used repeatedly to extend its validity on a dense subspace to all of $W_2^1(\Omega)$; this is a prototypical example of a **density argument** 稠密. ■

Remark 1.7.2 ((1.6.4)) *Note that* ***this proposition does not assert that pointwise values of*** u ***on*** $\partial\Omega$ ***make sense, only that*** $u|_{\partial\Omega}$ ***is square integrable on*** $\partial\Omega$. *This leaves open the possibility (cf. exercise 1.x.28) that u could be infinite at a dense set of points on $\partial\Omega$. For smooth functions,* ***the trace defined here is the same as the ordinary pointwise restriction to the boundary.***

Remark 1.7.3 ((1.6.5)) 不重要，没看懂。*The proposition, at first glance, says that functions in $W_2^1(\Omega)$ have boundary values in $L^2(\partial\Omega)$, and this is true. However, this by itself would not be a sharp result (not every element of $L^2(\partial\Omega)$ is the trace of some element of $W_2^1(\Omega)$). But on closer inspection it says something which is sharp, namely, that the $L^2(\partial\Omega)$ norm of a function can be bounded by just part of the $W_2^1(\Omega)$ norm (the square root of it), together with the $L^2(\Omega)$ norm. This result might seem strange until we see that it is dimensionally correct. That is, suppose that functions are measured in some unit U, and that L denotes the length unit. Then the units of the $W_2^1(\Omega)$ norm (ignoring lower order terms) equal U, and those of the $L^2(\Omega)$ norm equal $U \cdot L$. Neither of these matches the units of the square root of the left-hand side of (1.6.2), $U\sqrt{L}$, but the square root of their product does. Such a dimensionality argument can not prove an inequality such as (1.6.2), but it can be used to disprove one, or simplify its proof (cf. exercise 1.x.31).*

Now let us describe a generalization of Proposition 1.6.3 to more complex domains. One natural approach is to work in the class of Lipschitz domains. If $\partial\Omega$ is given as the graph of a function ϕ (cf. (1.4.4)), we can define the integral on $\partial\Omega$ as

$$\int_{\partial\Omega} f dS := \int_{\mathbb{R}^{n-1}} f(x,\phi(x))\sqrt{1+|\nabla\phi(x)|^2}dx.$$

If ϕ is Lipschitz, then the weight $\sqrt{1+|\nabla\phi(x)|^2}$ is an L^∞ function (cf. exercise 1.x.14). In this way (cf. Grisvard 1985), we can define the integral on any Lipschitz boundary, and correspondingly associated Lebesgue spaces. Moreover, the following result holds.

Theorem 1.7.4 ((1.6.6)) *Suppose that Ω has a Lipschitz boundary, and that p is a real number in the range $1 \le p \le \infty$. Then there is a constant, C, such that*

$$\boxed{\|v\|_{L^p(\partial\Omega)}} \le C\|v\|^{1-1/p}_{L^p(\Omega)}\|v\|^{1/p}_{W^1_p(\Omega)} \quad \forall v \in W_p^1(\Omega).$$

We will use the notation $\boxed{\hat{W}_p^1(\Omega)}$ to denote the subset of $W_p^1(\Omega)$, consisting of functions whose trace on $\partial\Omega$ is zero, that is

$$\hat{W}_p^1(\Omega) = \{v \in W_p^1(\Omega) : v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\}. \tag{1.6.7}$$

Similarly, we let $\hat{W}_p^k(\Omega)$ denote the subset of $W_p^k(\Omega)$ consisting of functions whose derivatives of order $k-1$ are in $\hat{W}_p^1(\Omega)$, i.e.

$$\hat{W}_p^k(\Omega) = \{v \in W_p^k(\Omega) : v^{(\alpha)}|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega) \quad \forall|\alpha| < k\}. \tag{1.6.8}$$

补充定义trace operator based on Harlim's note

Definition 1.7.5 (1.33) ***(Trace operator)*** *Let $v \in C(\bar{\Omega})$. We define the trace operator as the restriction of v on the boundary of Ω, that is, $\gamma_0(u) = u|_{\partial\Omega}$. This is a linear and continuous map.*

Theorem 1.7.6 (1.34) ***(Trace theorem)*** *Let Ω be a Lipschitz open subset of $\mathbb{R}^d$. The trace operator $\gamma_0 : C^1(\bar{\Omega}) \to C(\partial\Omega)$ can be extended to $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ and extended map is continuous,*

$$\|\gamma_0(v)\|_{L^2(\partial\Omega)} \le C\|v\|_{H^1(\Omega)},$$

for some $C > 0$ and any $v \in H^1(\Omega)$. Indeed $Ker(\gamma_0) = H_0^1(\Omega)$.

Remark 1.7.7 (1.35) *$\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is not onto. The range space $\gamma_0(H^1(\Omega)) := H^{1/2}(\partial\Omega) = \{v \in L^2(\partial\Omega) : v = \gamma_0(u)$ for some $u \in H^1(\Omega)\}$.*

Now we can generalize integration by parts formula in (I.2) to functions in Sobolev spaces.

Proposition 1.7.8 (1.36) *(Integration by parts) Let Ω be a Lipschitz open set in $\mathbb{R}^d$ and $u, v \in H^1(\bar{\Omega})$, then*

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)v(x)dx = -\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx + \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)n_i d\Gamma.$$

Proof. From (I.2), we have the identity to hold for $u, v \in C^1(\bar{\Omega})$. Take $u_n, v_n \in C^1(\bar{\Omega})$ such that $u_n \to u$ and $v_n \to v$ in $H^1(\Omega)$. This means $u_n \to u, v_n \to v$, $\partial_i u_n \to \partial_i u$, and $\partial_i v_n \to \partial_i v$ in $L^2(\Omega)$. Thus, we have $v_n \partial_i u_n \to v\partial_i u$ and $u_n \partial_i v_n \to u\partial_i v$ in $L^1(\Omega)$. By the continuity of the trace mapping theorem, we have $\gamma_0(u_n) \to \gamma_0(u)$ in $L^2(\partial\Omega)$ such that $\gamma_0(u_n)\gamma_0(v_n) \to \gamma_0(u)\gamma_0(v)$ in $L^1(\partial\Omega)$. Taking the limit of the integration by part formula in (I.2) for sequences of $u_n, v_n \in C^1(\bar{\Omega})$, we achieve the desired result. ■

Definition 1.7.9 (1.37) *(Trace of H^2 functions) Let Ω be a Lipschitz open subset of $\mathbb{R}^d$. For $u \in H^2(\Omega)$, we also define $\gamma_1(u) = \sum_{i=1}^d \gamma_0(\partial_i u)n_i$ where n_i denotes the component of normal vector $\vec{n}$. Indeed, $\gamma_1 : H^2(\Omega) \to L^2(\partial\Omega)$ is continuous. If $u \in C^2(\bar{\Omega})$, then $\gamma_1(u) = \nabla u \cdot \vec{n} = \frac{\partial u}{\partial \vec{n}}$, where the derivatives are defined in classical sense.*

With this definition, one can verify the Green's formula:

Proposition 1.7.10 (1.38) *For any $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega (\Delta u(x))v(x)dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x)dx + \int_{\partial\Omega} \gamma_1(u)\gamma_0(v)d\Gamma.$$

补充定义trace operator based on Harlim's note

1.8 Negative Norms and Duality

In this section we introduce ideas that lead to the definition of Sobolev spaces W_p^k for negative integers k. This definition is based on the concept of **duality in Banach spaces**. The **dual space**, B', to a Banach space, B, is a set of linear functionals on B. (A **linear functional** on a linear space B is simply a linear function from B into the reals, $\mathbb{R}$, i.e., a function $L : B \to \mathbb{R}$ such that

$$L(u + av) = L(u) + aL(v) \quad \forall u, v \in B, a \in \mathbb{R}.$$

More precisely, we distinguish between the linear space, B^*, of **all** linear functionals on B (cf. exercise 1.x.33), and the subspace $B' \subset B^*$ of **continuous** linear functionals on B. The following observation simplifies the characterization of such functionals.

Proposition 1.8.1 ((1.7.1)) *A linear functional, L, on a Banach space, B, is continuous if and only if it is bounded, i.e., if there is a finite constant C such that $|L(v)| \leq C\|v\|_B \quad \forall v \in B$.*

Proof. A bounded linear function is actually Lipschitz continuous, i.e.,

$$|L(u) - L(v)| = |L(u - v)| \leq C\|u - v\|_B \quad \forall u, v \in B.$$

Conversely, suppose L is continuous. If it is not bounded, then there must be a sequence $\{v_n\}$ in B such that $|L(v_n)|/\|v_n\|_B \geq n$. Renormalizing by setting $w_n = v_n/n\|v_n\|_B$ gives $|L(w_n)| \geq 1$ but $\|w_n\|_B \leq 1/n$, and thus $w_n \to 0$. But, by continuity of L, we should have $L(w_n) \to 0$, the desired contradiction. ∎

For a continuous linear functional, L, on a Banach space, B, the proposition states that the following quantity is always finite:

$$\|L\|_{B'} := \sup_{0 \neq v \in B} \frac{L(v)}{\|v\|_B}. \tag{1.7.2}$$

Exercise 1.x.34 shows that this **forms a norm on B', called the dual norm,** and one can show (cf. Trèves 1967) that B' **is also a Banach space.**

Example 1.8.2 ((1.7.3)) *The dual space of a Banach space need not be a mysterious object. One of the key results of Lebesgue integration theory is that the dual spaces of L^p can be easily identified, for $1 \leq p < \infty$. From Hölder's inequality, any function $f \in L^q(\Omega)$ (where $\frac{1}{q} + \frac{1}{p} = 1$ defines the dual index, q, to p) can be viewed as a continuous linear functional via*

$$L^p(\Omega) \ni v \to \int_\Omega v(x)f(x)dx.$$

One version of the Riesz Representation Theorem states that all continuous linear functionals on $L^p(\Omega)$ arise in this way, i.e., that $(L^p(\Omega))'$ is isomorphic to $L^q(\Omega)$.

Example 1.8.3 ((1.7.4)) *The dual space of a Banach space can also contain totally new objects. For example, Sobolev's inequality shows that the Dirac δ-function is a continuous linear functional on W_p^k, provided k and p satisfy the appropriate relation given in (1.4.6). Specifically, the Dirac δ-function is the linear functional*

$$W_p^k(\Omega) \ni v \to v(y) =: \delta_y(v),$$

where y denotes a given point in the domain Ω. It can be seen that this can not arise via an integration process using any locally integrable function (cf. exercise 1.x.36), i.e., it can not be viewed as a member of any of the spaces introduced so far.

Definition 1.8.4 ((1.7.5)) *Let p be in the range $1 \leq p \leq \infty$, and let k be a negative integer. Let q be the dual index to p, i.e., $\frac{1}{q} + \frac{1}{p} = 1$. Then the Sobolev space $W_p^k(\Omega)$ is defined to be the dual space $(W_q^{-k}(\Omega))'$ with norm given by the dual norm (cf. (1.7.2)).*

Remark 1.8.5 ((1.7.6)) *Note that we have defined the negative-index Sobolev spaces so that, if the same definition were used as well for $k = 0$, then the two definitions (cf. (1.3.1)) would agree for $1 < p < \infty$, in view of Remark 1.7.3. Note also that different dual spaces are used to define negative-index Sobolev spaces, in particular, it is frequently useful to use the dual of a subspace of $W_q^{-k}(\Omega)$, and in view of exercise 1.x.35, this leads to a slightly larger space. In either case, the negative Sobolev spaces are big enough to include interesting new objects, such as the Dirac δ-function. Example 1.7.4 shows that $\delta \in W_p^k(\Omega)$ provided $k < -n + n/p$ (or $k \leq -n$ if $p = \infty$).*

补充定义同济PDE中对偶积的定义，$L(v)$是什么？或者补充Harlim's note using $\langle L, v \rangle$

1.8.1 广义函数简介

广义函数是研究偏微分方程的重要工具之一。为了清楚和正确地理解基本解的概念与物理意义，有必要简单地介绍一些广义函数的基本知识。

广义函数作为一门严格的数学理论，是与Sobolev 和Schwartz 的名字紧密联系在一起的。前者扩充了微商的概念，从而为偏微分方程广义解的研究奠定了基础。后者在局部凸线性拓扑空间的基础上，确立了广义函数的严格数学理论。为了理解和掌握这个理论，必须具备一定的泛函分析基础，而且亦提出了本课程的讨论范围。因此，在这里我们将着重介绍δ 函数及运算法则，并尽可能叙述得直观一些。为简单起见，下面的讨论都是以一元广义函数为例，事实上对于多元的情形，讨论是完全类似的。

历史上，广义函数的引入是出于描述物理现象的需要。Dirac 函数，即δ 函数是最常用的广义函数，被认为具有如下性质：

$$\delta(x)=\begin{cases}0, & \text{当}x\neq 0\text{ 时},\\ \infty, & \text{当}x=0\text{ 时},\end{cases} \tag{1.30}$$

$$\int_{-\infty}^{\infty}\delta(x)\,dx=1. \tag{1.31}$$

它是描述集中分布的量（如点电荷、点热源、瞬时脉冲、集中质量等）的有力工具。例如，在直线上除$x=\xi$ 处有一个单位点电荷外，其余各处均无电荷，用线电荷密度$\rho(x)$ 来描述这样的电荷分布，可得

$$\rho(x)=\begin{cases}0, & \text{当}x\neq \xi\text{ 时},\\ \infty, & \text{当}x=\xi\text{ 时}.\end{cases} \tag{1.32}$$

注意(1.32) 规定的$\rho(x)$ 虽能表明在$x=\xi$ 处有一个点电荷，但不能表明其电量的大小。由于这时在直线上分布的总电量等于1 个单位，因此，还应要求$\rho(x)$ 满足

$$\int_{-\infty}^{\infty}\rho(x)\,dx=1. \tag{1.33}$$

用δ 函数来表示由(1.32) 和(1.33) 规定的$\rho(x)$ 就有

$$\rho(x)=\delta(x-\xi). \tag{1.34}$$

显然，δ 函数和$\rho(x)$ 不是通常意义的函数，它们的积分运算在分析中也不是人们所能接受的。因此，为了刻画这类“奇异函数”及其运算，需要我们扩充函数的概念，扩充的方法之一就是把它们与线性泛函联系起来。

我们知道，泛函是从函数到数的对应，其中作为“自变量”的是函数，它的变化范围是泛函的定义集合。因此，首先来讨论广义函数这种泛函的定义集合，即基本空间。与第一章§2 所引进的函数集合$C_0^\infty(\Omega)$ 相类似，以$C_0^\infty(\mathbb{R})$ 表示所有当$|x|$ 充分大时恒等于零的无穷次连续可微函数$\varphi(x)$ 构成的集合。

Definition 1.8.6 (1.3) *如果$\varphi(x)\in C_0^\infty(\mathbb{R})$, $\varphi_n(x)\in C_0^\infty(\mathbb{R})$, $n=1,2,\cdots$, 且*

1. *存在$M>0$, 使得当$|x|\geq M$ 时$\varphi(x)\equiv 0$, $\varphi_n(x)\equiv 0$, $n=1,2,\cdots$,*

2.
$$\lim_{n\to\infty}\max_{[-M,M]}|\varphi_n(x)-\varphi(x)|=0,$$
$$\lim_{n\to\infty}\max_{[-M,M]}|\varphi_n^{(k)}(x)-\varphi^{(k)}(x)|=0,\quad k=1,2,\cdots,$$

则称序列$\{\varphi_n\}$ 收敛于φ。规定了上述收敛性的线性空间$C_0^\infty(\mathbb{R})$ 称为**基本空间** $\mathcal{D}(\mathbb{R})$。$\varphi\in\mathcal{D}(\mathbb{R})$ 称为**试验函数**。

现在可以定义广义函数f。

Definition 1.8.7 (1.4) 如果$f:\mathcal{D}(\mathbb{R})\to\mathbb{R}$ 是线性连续泛函，则称f 是一个**广义函数**。设$\varphi\in\mathcal{D}(\mathbb{R})$ 是一个试验函数，用$\langle f,\varphi\rangle$ 表示它所对应的数值，称为**对偶积**。

因此，按照定义1.4，广义函数f 也可记成$f:\varphi\to\langle f,\varphi\rangle$。在定义1.4 中，线性是指：对于任意实数$a,b$ 及任意试验函数φ,ψ 成立

$$\langle f,a\varphi+b\psi\rangle=a\langle f,\varphi\rangle+b\langle f,\psi\rangle;$$

连续性是指：对于任意试验函数序列$\{\varphi_n\}$ 和试验函数φ，只要序列φ_n 在$\mathcal{D}(\mathbb{R})$ 意义下（即按照定义1.3 中规定的意义）收敛于φ，则有

$$\lim_{n\to\infty}\langle f,\varphi_n\rangle=\langle f,\varphi\rangle.$$

所有广义函数组成的集合记作$\mathcal{D}'(\mathbb{R})$，**分布空间**。

Example 1.8.8 (1) 使任意试验函数$\varphi(x)$ 与它在$x=0$ 处的函数值$\varphi(0)$ 对应的广义函数称为δ 函数，即δ 函数满足

$$\langle\delta,\varphi\rangle=\varphi(0),\quad\forall\varphi\in\mathcal{D}(\mathbb{R}).\tag{1.35}$$

事实上，容易验证$\varphi\to\varphi(0)$ 是$\mathcal{D}(\mathbb{R})$ 上的线性连续泛函。这就是前面所说的δ 函数的严格定义。

Example 1.8.9 (2) 设$f(x)$ 是$\mathbb{R}$ 上的局部绝对可积函数，即在任何有限区间(a,b) 上积分

$$\int_a^b|f(x)|\,dx$$

存在，则试验函数$\varphi(x)$ 与积分值

$$\int_{-\infty}^{\infty}f(x)\varphi(x)\,dx$$

的对应是一个广义函数。

注意对偶积

$$\langle f,\phi\rangle=\int_{-\infty}^{\infty}f(x)\varphi(x)dx$$

的线性性质易证，但其连续性的证明较复杂，此处略去。

由例2可见，任一局部绝对可积函数$f(x)$，可按照它与试验函数φ的乘积的积分确定一个广义函数，我们把后者与函数$f(x)$ 不加区分，仍记为f。我们用$L_{\text{loc}}(\mathbb{R})$ 表示所有在$\mathbb{R}$ 上局部绝对可积的函数组成的集合，例2 表明

$$L_{\text{loc}}(\mathbb{R})\subseteq\mathcal{D}'(\mathbb{R}).\tag{1.36}$$

可以证明δ 函数不是由$L_{\text{loc}}(\mathbb{R})$ 中函数产生的广义函数，也就是说，与(1.36) 相反的包含关系不成立（证明见附注）。因此，广义函数是局部绝对可积函数类的扩充。

Remark 1.8.10 附注 δ 函数不是局部绝对可积函数。事实上假若不然，存在$f(x)\in L_{loc}(\mathbb{R})$，使得$\forall\varphi\in\mathcal{D}(\mathbb{R})$，有

$$\int_{-\infty}^{\infty}f(x)\varphi(x)\,dx=\varphi(0),$$

取$\varphi_n(x)=\rho_n(x,0)=n^2\rho(nx,0)$，其中$\rho(x,y)$ 的定义见第一章§2，由可积函数的绝对连续性，则当n 充分大时，

$$\varphi_n(0)=\left|\int_{-\infty}^{\infty}f(x)\varphi_n(x)\,dx\right|\le\varphi_n(0)\int_{-\frac{1}{n}}^{\frac{1}{n}}|f(x)|\,dx<\varphi_n(0),$$

于是得到矛盾。由此可知广义函数是局部绝对可积函数的推广。

广义函数可以按照通常的方式定义加法与数乘运算，即若$f,g\in\mathcal{D}'(\mathbb{R})$，则$f+g\in\mathcal{D}'(\mathbb{R})$，且

$$\langle f+g,\varphi\rangle=\langle f,\varphi\rangle+\langle g,\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\mathbb{R}),$$

若$f\in\mathcal{D}'(\mathbb{R})$，$a\in\mathbb{R}$，则$af\in\mathcal{D}'(\mathbb{R})$，且

$$\langle af,\varphi\rangle=a\langle f,\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\mathbb{R}).$$

Remark 1.8.11 附注 一般而言，广义函数与广义函数的乘积是没有意义的。

现在来定义广义函数序列的极限：

Definition 1.8.12 (1.5) 设$f\in\mathcal{D}'(\mathbb{R})$，$\{f_n\}\subseteq\mathcal{D}'(\mathbb{R})$，如果

$$\lim_{n\to\infty}\langle f_n,\varphi\rangle=\langle f,\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\mathbb{R}),\tag{1.37}$$

则称广义函数f 是广义函数序列$\{f_n\}$ 当$n\to\infty$ 时的极限，记作

$$f_n\to f\quad(\text{在}\mathcal{D}'(\mathbb{R})\text{ 的意义下}).\tag{1.38}$$

类似可以定义依赖于参数的广义函数的极限：

Definition 1.8.13 (1.6) 设$f\in\mathcal{D}'(\mathbb{R})$，$f_\varepsilon\in\mathcal{D}'(\mathbb{R})$ $(\varepsilon>0)$，如果

$$\lim_{\varepsilon\to0}\langle f_\varepsilon,\varphi\rangle=\langle f,\varphi\rangle,\quad\forall\varphi\in\mathcal{D}(\mathbb{R}),$$

则称广义函数f 是广义函数族$\{f_\varepsilon\}$ 当$\varepsilon\to0$ 时的极限，记作

$$f_\varepsilon\to f\qquad(\varepsilon\to0,\quad\text{在}\mathcal{D}'(\mathbb{R})\text{的意义下})$$

1.8.2 Dual in Harlim's note

Definition 1.8.14 (1.31) *(Dual of H_0^1) Let*

$$H^{-1}(\Omega):=\{T\in\mathcal{D}'(\Omega)\text{ such that }|\langle T,\varphi\rangle|\le C|\varphi|_{H^1(\Omega)}\text{ for some }C\text{ and }\forall\varphi\in\mathcal{D}(\Omega)\},$$

equipped with the following norm,

$$\|T\|_{H^{-1}(\Omega)}:=\sup_{\varphi\in H_0^1(\Omega)}\frac{\langle T,\varphi\rangle}{|\varphi|_{H^1}}.$$

Remark 1.8.15 (1.32) *We have the following properties:*

1. *This is a space of distributions of Ω that are continuous with respect to the seminorm in $[1,6]$, which is a norm of $H_0^1(\Omega)$. More precisely, $H^{-1}(\Omega)=(H_0^1(\Omega))^*$.*

2. *Let* $f \in L^2(\Omega)$*, then* $\boxed{\partial_i f \in H^{-1}(\Omega)}$ *and* $\langle \partial_i f, \varphi \rangle = -\int_\Omega f \partial_i \varphi$ *for all* $\varphi \in \mathcal{D}(\Omega)$*. To see this, by definition* $\partial_i f \in \mathcal{D}'(\Omega)$ *since it is a distributional partial derivative. Since* $f \in L^1_{loc}$*, we can employ (I.3) to deduce the identity* $\langle \partial_i f, \varphi \rangle = -\int_\Omega f \partial_i \varphi$ *for all* $\varphi \in \mathcal{D}(\Omega)$*. Finally, by Cauchy-Schwartz inequality,*

$$\boxed{|\langle \partial_i f, \varphi \rangle| \leq \|f\|_{L^2(\Omega)} \|\partial_i \varphi\|_{L^2(\Omega)} \leq C|\varphi|_{H^1(\Omega)}}$$

where $C = \|f\|_{L^2(\Omega)}$*. This means* $\boxed{\partial_i f \in H^{-1}(\Omega)}$ *and* $\boxed{\|\partial_i f\|_{H^{-1}(\Omega)} \leq C = \|f\|_{L^2(\Omega)}}$.

3. *The result [ii.] above means* $\partial_i : L^2(\Omega) = H^0(\Omega) \to H^{-1}(\Omega)$ ***is linear continuous from*** $L^2(\Omega)$ *to* $H^{-1}(\Omega)$*. One can also show that* $-\Delta$ ***is a linear continuous from*** $\boxed{H_0^1(\Omega)}$ *to* $\boxed{H^{-1}(\Omega)}$*. Also, for all* $u, v \in H_0^1(\Omega)$*, we have*

$$\langle -\Delta u, v \rangle = \int_\Omega \nabla u \cdot \nabla v dx. \tag{1.2}$$

To see this identity, it is clear that if $u \in H_0^1$*, then* $\partial_i u \in L^2(\Omega)$*, so by [ii.] we have* $\langle -\partial_i u, \varphi \rangle = \int_\Omega \partial_i u \partial_i \varphi$ *for all* $\varphi \in \mathcal{D}(\Omega)$ *and that*

$$|\langle -\partial_i u, \varphi \rangle| \leq \|\partial_i u\|_{L^2(\Omega)} |\varphi|_{H^1(\Omega)},$$

so it is an element of $H^{-1}(\Omega)$*. Furthermore, for some* $C > 0$*,*

$$|\langle -\Delta u, \varphi \rangle| \leq \sum_{i=1}^d |\langle -\partial_i u, \varphi \rangle| \leq C|\varphi|_{H^1(\Omega)},$$

so $-\Delta u \in H^{-1}(\Omega) = (H_0^1)^*$*. Now, take* $v \in H_0^1(\Omega)$*. Since* $\mathcal{D}(\Omega)$ *is dense in* $H_0^1(\Omega)$*, there exist* $v_n \in \mathcal{D}(\Omega)$ *such that* $v_n \to v$ *in* H_0^1*. By continuity of* $-\Delta u$*, we have*

$$|\langle -\Delta u, v_n \rangle - \langle -\Delta u, v \rangle| \leq \| -\nabla u\|_{L^2(\Omega)} |v_n - v|_{H^1(\Omega)} \to 0.$$

On the other hand, by [ii.]

$$\langle -\Delta u, v_n \rangle = \int_\Omega \nabla u \cdot \nabla v_n \to \int_\Omega \nabla u \cdot \nabla v$$

since $\nabla v_n \to \nabla v$ *in* $L^2(\Omega)$*, and* $\nabla u \cdot \nabla v_n \to \nabla u \cdot \nabla v$ *in* $L^1(\Omega)$*. From (I.8) and (I.9), we achieved the identity in (I.7).*

补充定义PDE中对偶积的定义，$L(v)$是什么？或者补充Harlim's note using $\langle L, v \rangle$