MATH1312: Lecture Note on Probability Theory and Mathematical Statistics

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2024 年 11 月 11 日

Contents

Chapter 1

Characteristic Functions

See Chapter 4 of [1] for reference.

1.1 Properties of Characteristic Functions

Let X be a random variable and $F(x)$ be its (cumulative) distribution function (CDF).

Definition 1.1.1 *The function*

$$\phi(t) = E[e^{itX}],$$

is called the characteristic function (Ch.f.) of random variable X or of the CDF $F(x)$.

1.1.1 Case I. Discrete random variable

Let X take on values of x_k $(k = 1, 2, \ldots)$ with probability

$$P(X = x_k) = p_k, \quad \text{with } \sum_{k=1}^{\infty} p_k = 1.$$

Then the Ch.f. of X is

$$\phi(t) = E[e^{itX}] = \sum_{k=1}^{\infty} p_k e^{itx_k}.$$

（比较判别法）

Remark 1.1.2 *Since $|\phi(t)| \leq \sum_k \left|p_k e^{itx_k}\right| = \sum_k p_k = 1 < \infty$, this is absolutely convergent. At the same time, based on Weierstrass Criterion, we also have that $\phi(t)$ is uniformly convergent!*

Remark 1.1.3 *The Ch.f. $\phi(t)$, as the sum of a uniformly convergent series of continuous functions, is **continuous**, for every real value of t.*

Example 1.1.4 *Let*

$$\begin{aligned} x_1 &= -1, \quad P(X = -1) = 0.5, \\ x_2 &= +1, \quad P(X = +1) = 0.5. \end{aligned}$$

Then the Ch.f. can be computed as

$$\phi(t) = 0.5e^{-it} + 0.5e^{it} = 0.5(\cos t - i \sin t) + 0.5(\cos t + i \sin t) = \cos t.$$

1.1.2 Case II. Continuous random variable

Let X be a continuous random variable with density function $f(x)$. Then the Ch.f. is

$$\phi(t) = E[e^{itX}] = \int_{-\infty}^{\infty} f(x)e^{itx}dx.$$

Remark 1.1.5 *Since* $|\phi(t)| \leq \int_{-\infty}^{\infty} f(x)\left|e^{itx}\right| dx = \int_{-\infty}^{\infty} f(x)dx = 1 < \infty$, *the integral is absolutely and uniformly convergent (which means that one can interchange the order of limit and integral,* $\lim_{t\to t_0}\int = \int \lim_{t\to t_0}$*). Hence* $\phi(t)$ *is a **continuous** function for all* t.

Example 1.1.6 *Consider the random variable* X *with density*

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x \leq 1 \\ 0, & x > 1. \end{cases}$$

Then

$$\phi(t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx = \int_0^1 e^{itx}dx = \left[\frac{e^{itx}}{it}\right]|_0^1 = \frac{e^{it}-1}{it}, \ t \neq 0.$$

As $t = 0$, $\phi(0) = \int_0^1 dx = 1$. *We now show the continuity at* $t = 0$,

$$\lim_{t\to 0}\phi(t) = \lim_{t\to 0}\frac{e^{it}-1}{it} = \lim_{t\to 0}\frac{ie^{it}}{i} = 1,$$

where L'Hospital was used. Thus, $\phi(t)$ *is continuous as expected.*

1.1.3 properties of ch.f.

We can easily show the following properties:

(1) $\phi(0) = Ee^0 = E1 = 1$.

(2) Since $|\phi(t)| = \left|E[e^{itX}]\right| \leq E|e^{itX}| = 1$, we have

$$|\phi(t)| \leq 1 = \phi(0).$$

(3) Since we compute

$$\begin{aligned} \phi(-t) &= E[e^{-itX}] = E(\cos tX - i\sin tX) = E(\cos tX) - iE(\sin tX), \\ \phi(t) &= E[e^{itX}] = E(\cos tX + i\sin tX) = E(\cos tX) + iE(\sin tX), \end{aligned}$$

we have

$$\phi(-t) = \overline{\phi(t)}.$$

Q: Now the question is that every Ch.f. satisfies above conditions, however, the conditions are not sufficient. That is, not every function $\phi(t)$ satisfying these conditions (1)-(3) is a Ch.f. of some random variable.

Remark 1.1.7 *See Marcinkiewicz in [1]. A function* $\phi(t)$, *which is NOT identically constant and which, in a neighborhood of zero, can be represented in*

$$\phi(t) = 1 + O(t^{2+\alpha}),$$

with $\alpha > 0$ *cannot be a Ch.f.*

Example 1.1.8 *Functions* $\phi(t) = e^{-t^4}, \phi(t) = \frac{1}{1+t^4}$ *cannot be Ch.f.*

Next, we provide a sufficient and necessary condition for a function being Ch.f without detailed proof.

Theorem 1.1.9 *(Bochner's Theorem) Let the function* $\phi(t)$ *defined for* $-\infty < t < \infty$ *satisfy condition* $\phi(0) = 1$*. The function* $\phi(t)$ *is the ch.f. of some distribution function if and only if*
1. $\phi(t)$ *is continuous.*
2. for $n = 1, 2, 3, \ldots$*, and every real* $t_1, \ldots, t_n$ *and complex* $a_1, \ldots, a_n$*, we have*

$$\sum_{j,k=1}^{\infty} \phi(t_j - t_k) a_j \overline{a}_k \geq 0,$$

i.e., $\phi(t)$ *satisfies the (symmetric) positive-definiteness.*

The proof is hard. Thus we only give a sketch of the proof for one easy case.

Proposition 1.1.10 *If* g *on a given Domain is positive-definite, then for every* $u \in$ *Domain,*

$$g(0) \geq 0, \quad g(-u) = \overline{g}(u), \quad |g(u)| \leq g(0).$$

Moreover, if g *is continuous at the origin, then* g *is uniformly continuous on the set of limit points of Domain. (In this sense,* $g(t)$ *is identical to a Ch.f.* $\phi(t)$ *up to a constant.)*

Proof. (1) Take $\{t_1 = 0\}$, then

$$g(0)\alpha_1\bar{\alpha}_1 \geq 0 \Rightarrow g(0) \geq 0.$$

(2) Take $\{t_1 = 0, t_2 = u\}$, then

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} g(0) & g(u) \\ g(-u) & g(0) \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{bmatrix} \geq 0.$$

Since g is positive-definite, this implies that g is Hermitian, that is,

$$g(-u) = \overline{g}(u).$$

This can also be found from the following,

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} g(0) & g(u) \\ g(-u) & g(0) \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{bmatrix} = |\alpha_1|^2 + g(u)\alpha_1\bar{\alpha}_2 + g(-u)\bar{\alpha}_1\alpha_2 + |\alpha_2|^2 \geq 0.$$

This says that $g(u)\alpha_1\bar{\alpha}_2 + g(-u)\bar{\alpha}_1\alpha_2$ is real for any complex α_1, α_2. First, we have

$$\text{Im}(g(-u)\bar{\alpha}_1\alpha_2) = -\text{Im}\left[g(u)\alpha_1\bar{\alpha}_2\right] = \text{Im}\left[\overline{g(u)\alpha_1\bar{\alpha}_2}\right] = \text{Im}\left[\overline{g}(u)\bar{\alpha}_1\alpha_2\right],$$

which indicates that $\text{Im}(g(-u)) = \text{Im}\left[\overline{g}(u)\right]$. Second, take $\alpha_1 = i, \alpha_2 = 1$, then we have $g(u)i - g(-u)i$ is real, which indicates that $\text{Re}\left[g(-u)\right] = \text{Re}\left[g(u)\right]$. Thus we have

$$g(-u) = \overline{g}(u)$$

for any u.

(3) The determinant of above matrix is nonnegative,

$$|g(0)|^2 \geq g(u)g(-u) = |g(u)|^2 .$$

(4) If $g(0) = 0$, based on $|g(u)| \leq |g(0)| = 0 \Rightarrow g(u) \equiv 0$, which is trivial case. Let $g(0) \neq 0$, we can always let $h(u) = g(u)/g(0)$ to normalize the function, so that $h(0) = 1$.
(5) Now g is continuous at the origin. We have

$$\begin{matrix} 0 \\ u \\ u' \end{matrix} \begin{bmatrix} g(0) & g(-u) & g(-u') \\ g(u) & g(0) & g(u-u') \\ g(u') & g(u'-u) & g(0) \end{bmatrix}.$$

The determinant is

$$\begin{aligned} \det &= 1 + \overline{g}(u)g(u')g(u-u') + g(u)\overline{g}(u')\overline{g}(u-u') \\ &\quad - |g(u')|^2 - |g(u)|^2 - |g(u-u')|^2 \geq 0. \end{aligned}$$

Then we simplify

$$\begin{aligned} |g(u')|^2 + |g(u)|^2 &\leq 1 - |g(u-u')|^2 + \text{Re}\left[g(u)\overline{g}(u')\overline{g}(u-u')\right], \\ |g(u')|^2 + |g(u)|^2 - 2\text{Re}\left[g(u)\overline{g}(u')\right] &\leq 1 - |g(u-u')|^2 + \text{Re}\left[g(u)\overline{g}(u')\overline{g}(u-u')\right] - 2\text{Re}\left[g(u)\overline{g}(u')\right], \\ |g(u) - g(u')|^2 &\leq 1 - |g(u-u')|^2 - 2\text{Re}\left[g(u)\overline{g}(u')(1-\overline{g}(u-u'))\right]. \end{aligned}$$

Since g is continuous at 0, that is, $g(u-u') \to g(0) = 1$ as $u - u' \to 0$, then

$$|g(u) - g(u')|^2 \leq 1 - 1 - 0 = 0, \qquad \text{as } u - u' \to 0,$$

i.e., $g(u) - g(u') \to 0$, which is uniformly continuous. ■

Remark 1.1.11 *g coincides on Domain with a Ch.f. up to a multiplicative constant.*

Lemma 1.1.12 *(Herglotz Lemma or Discrete Bochner Theorem) A function g on the set $D_s = \{\ldots, -2c, -c, 0, +c, +2c, \ldots\}$ is positive-definite if and only if it coincides on this set with a ch.f. $f(u) = \int_{-\pi/c}^{\pi/c} e^{iux} dF(x)$, i.e., $f(ck) = \int_{-\pi/c}^{\pi/c} e^{ickx} dF(x)$, for all integers k.*

Proof. Assume that $c > 0$. For every integer n and every finite x, we have positive-definite g,

$$\frac{1}{2\pi n} \sum_{h=1}^{n} \sum_{j=1}^{n} g((j-h)c) e^{-i(j-h)x} \geq 0.$$

Let $k = j - h$, then the **density** function

$$\begin{aligned} p_n(x) &= \frac{1}{2\pi n} \sum_{k=-n+1}^{n-1} (n - |k|) g(kc) e^{-ikx} \geq 0 \\ &= \frac{1}{2\pi} \sum_{k=-n+1}^{n-1} \underbrace{(1 - \frac{|k|}{n}) g(kc)}_{a_k, \text{ Fourier coefficient}} \; e^{-ikx} \geq 0, \end{aligned}$$

where we notice that when $k = n-1$ only $(j,h) = (n,1)$, and when $k = n-2$ there is $(j,h) = (n,2),(n-1,1)$, and so on. We now take the Fourier inverse transform, that is, multiplying by e^{ikx} and then taking the integral for x from $-\pi$ to π,

$$\begin{aligned}\underbrace{a_k}_{\propto g} &= \underbrace{\left(1-\frac{|k|}{n}\right)}_{\to 1} g(kc) = \int_{-\pi}^{\pi} e^{ikx} \underbrace{p_n}_{\geq 0, pdf}(x)dx \\ &\underbrace{=}_{x=cy} \int_{-\pi/c}^{\pi/c} e^{ikcy} p_n(cy)cdy = \int_{-\pi/c}^{\pi/c} e^{ikcy} d\underbrace{P_n(cy)}_{CDF}.\end{aligned}$$

Let $n \to \infty$,

$$g(u) \sim f(u) = \int_{-\pi/c}^{\pi/c} e^{iux} dF(x),$$

where $P_n(cx) \to F(x)$ as $n \to \infty$. ■

Let us go back to the half part of the proof of Bochner's Theorem: let the function $\phi(0) = 1$. The function ϕ on $\mathbb{R}$ is a character function of some distribution function if and only if it is continuous and positive-definite.

Proof. We only prove that ch.f. $\Rightarrow$ positive-definite. Let $\phi(t)$ be the ch.f. and $F(x)$ be the CDF. Then using the definition $\phi(t) = \int_{\mathbb{R}} e^{itx} dF(x)$, we have

$$\begin{aligned}\sum_{j,k} \phi(t_j - t_k) a_j \bar{a}_k &= \int_{\mathbb{R}} \sum_{j,k} e^{i(t_j - t_k)x} a_j \bar{a}_k dF(x) \\ &= \int_{\mathbb{R}} |\sum_j e^{it_j x} a_j|^2 dF(x) \geq 0.\end{aligned}$$

The opposite direction, ch.f. $\Leftarrow$ positive-definite + continuous, of the proof is difficult. ■

Remark 1.1.13 *A function ϕ on $\mathbb{R}$ is positive-definite and Lebesque-measurable if and only if it coincides a.e. with a Ch.f.*

Remark 1.1.14 *The continuity at at least one point is necessary.*

1.2 Characteristic Functions and Moments

1.2.1 relation between Ch.f. and moments

Suppose ℓth moment of a random variable X, $m_\ell = EX^\ell$ exists, i.e., $E|X^\ell| < \infty$. Then

Discrete: Since

$$\sum_k |i^\ell x_k^\ell p_k e^{itx_k}| = \sum_k |x_k^\ell p_k| = E|X^\ell| < \infty,$$

we can differentiate ℓ times of the Ch.f. $\phi(t)$,

$$\phi^{(\ell)}(t) = \sum_k i^\ell x_k^\ell p_k e^{itx_k} = E(i^\ell X^\ell e^{itX}),$$

where we can interchange the order of the summation and the differentiation due to the uniform convergence.

Continuous: Similarly, since for the density $f(x)$, we have

$$\int_{\mathbb{R}} |i^\ell x^\ell f(x) e^{itx}| dx = \int_{\mathbb{R}} |x^\ell f(x)| dx = E|X^\ell| < \infty,$$

thus we obtain

$$\phi^{(\ell)}(t) = \int_{\mathbb{R}} i^\ell x^\ell f(x) e^{itx} dx = E(i^\ell X^\ell e^{itX}).$$

Therefore, we conclude from above that

$$\begin{aligned} \phi^{(\ell)}(0) &= i^\ell E X^\ell = i^\ell m_\ell, \\ m_\ell &= \frac{\phi^{(\ell)}(0)}{i^\ell}. \end{aligned} \tag{1.1}$$

Theorem 1.2.1 *If the ℓth moment m_ℓ of a random variable exists, it is expressed by equation (1.1), where $\phi^{(\ell)}(0)$ is the ℓth derivative of the Ch.f. $\phi(t)$ of this random variable at $t = 0$.*

Example 1.2.2 *Let X be a Poisson distribution with*

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

The ch.f. is

$$\begin{aligned} \phi(t) &= \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{it}\lambda)^k}{k!} \\ &= \exp(-\lambda) \exp(\lambda e^{it}) = \exp[\lambda(e^{it} - 1)]. \end{aligned}$$

One can compute the derivative,

$$\begin{aligned} \phi'(t) &= \lambda i \exp(it) \exp[\lambda(e^{it} - 1)], \\ m_1 &= \frac{\phi'(0)}{i} = \frac{\lambda i}{i} = \lambda. \end{aligned}$$

Similarly,

$$\begin{aligned} \phi''(t) &= -\lambda \exp(it) \exp[\lambda(e^{it} - 1)] + (\lambda i \exp(it))^2 \exp[\lambda(e^{it} - 1)] \\ m_2 &= \frac{\phi''(0)}{i^2} = \frac{-\lambda - \lambda^2}{-1} = \lambda^2 + \lambda. \end{aligned}$$

Thus, the variance is

$$\sigma^2 = m_2 - (m_1)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Example 1.2.3 *For normal distribution with density*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

we have the Ch.f.

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-it)^2/2} e^{-\frac{t^2}{2}} dx = e^{-\frac{t^2}{2}}.$$

Since

$$\begin{aligned} \phi'(t) &= -t e^{-\frac{t^2}{2}} \Rightarrow m_1 = \frac{\phi'(0)}{i} = 0. \\ \phi''(t) &= t^2 e^{-\frac{t^2}{2}} - e^{-\frac{t^2}{2}} \Rightarrow m_2 = \frac{\phi''(0)}{i^2} = \frac{-1}{-1} = 1. \end{aligned}$$

Moreover,

$$m_{2k+1} = 0, \quad m_{2l} = (2l - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2l - 1).$$

Remark 1.2.4 *An example of a random variable whose expectation does not exist and whose Ch.f. is differentiable at $t = 0$ is given in Problem 4.9 of Fisz book.*

Remark 1.2.5 *If the Ch.f. $\phi(t)$ has a finite derivative of an even order $2k$ at $t = 0$, then the moment of order $2k$ of the corresponding random variable exists (Problem 4.8 of Fisz book). In this case, all the moments of order smaller than $2k$ exist.*

1.2.2 Linear transformation

Translation.

$$Y = X + b.$$

Then the Ch.f. is

$$\phi_Y(t) = E(e^{itY}) = E(e^{it(X+b)}) = e^{itb}\phi_X(t).$$

Scalar Multiplication.

$$Y = aX.$$

Then the Ch.f. is

$$\phi_Y(t) = E(e^{itY}) = E(e^{itaX}) = \phi_X(at).$$

In particular, if $a = -1$,

$$\phi_Y(t) = \phi_X(-t) = \overline{\phi_X(t)}.$$

Linear transformation.

$$\begin{aligned} Y &= aX + b. \\ \phi_Y(t) &= e^{itb}\phi_X(at). \end{aligned}$$

In particular, let m_1 be the mean and σ be the standard deviation,

$$Y = \frac{X - m_1}{\sigma},$$

then

$$\phi_Y(t) = \exp\left(-\frac{m_1 it}{\sigma}\right)\phi_X(\frac{t}{\sigma}).$$

1.3 Semi-invariants

Sometimes it is convenient to deal with a set of parameters other than the set of moments. Consider

$$\psi(t) = \log\phi(t),$$

where $\phi(t)$ is the Ch.f. Let us expand $\phi(t)$ in a neighborhood of $t = 0$,

$$\phi(t) = 1 + \sum_{s=1}^{\infty}\frac{m_s}{s!}(it)^s := 1 + z. \tag{1.2}$$

Then

$$\psi(t) = \log\phi(t) = \log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{s=1}^{\infty}\frac{\kappa_s}{s!}(it)^s. \tag{1.3}$$

Definition 1.3.1 *The coefficients κ_s in above (1.3) are called semi-invariants.*

By combining (1.2) and (1.3), one can further observe that

$$\begin{aligned}\phi(t) &= 1+\sum_{s=1}^{\infty}\frac{m_s}{s!}(it)^s = \exp\left(\psi(t)\right) = \exp\left[\sum_{s=1}^{\infty}\frac{\kappa_s}{s!}(it)^s\right] \\ &= 1+\sum_{s=1}^{\infty}\frac{\kappa_s}{s!}(it)^s + \frac{1}{2!}\left[\sum_{s=1}^{\infty}\frac{\kappa_s}{s!}(it)^s\right]^2 + \frac{1}{3!}\left[\sum_{s=1}^{\infty}\frac{\kappa_s}{s!}(it)^s\right]^3 + \cdots \qquad (1.4)\end{aligned}$$

By comparing (1.4), we have

$$\begin{aligned}\kappa_1 &= m_1, \\ \kappa_2 &= m_2 - m_1^2 = \sigma^2, \\ \kappa_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\ \kappa_4 &= m_4 - 3m_2^2 - 4m_1m_3 + 12m_1^2m_2 - 6m_1^4.\end{aligned}$$

Also,

$$\begin{aligned}m_1 &= \kappa_1, \\ m_2 &= \kappa_2 + \kappa_1^2, \\ m_3 &= \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3, \\ m_4 &= \kappa_4 + 3\kappa_2^2 + 4\kappa_1\kappa_3 + 6\kappa_1^2\kappa_2 + \kappa_1^4.\end{aligned}$$

Example 1.3.2 *Compute the semi-invariants of the Poisson distribution.*

$$\begin{aligned}\phi(t) &= \exp[\lambda(e^{it}-1)], \\ \psi(t) &= \log\phi(t) = \lambda(e^{it}-1) = \lambda\left(\sum_{k=0}^{\infty}\frac{(it)^k}{k!} - 1\right) = \lambda\sum_{k=1}^{\infty}\frac{(it)^k}{k!}.\end{aligned}$$

Thus,

$$\kappa_s = \lambda, \quad (s = 1, 2, \ldots).$$

1.4 The characteristic function of the sum of independent random variables

Let X and Y be two independent random variables. Find the Ch.f. of

$$Z = X + Y.$$

Let ϕ, ϕ_1, ϕ_2 denote the respective Ch.f. of Z, X, Y,

$$\phi(t) = Ee^{itZ} = Ee^{it(X+Y)} \overset{\text{independence}}{=} Ee^{itX}Ee^{itY} = \phi_1(t)\phi_2(t).$$

Theorem 1.4.1 *The Ch.f. of the sum of an arbitrary finite number of independent random variables equals the product of their Ch.f., i.e.,*

$$Z = X_1 + \cdots + X_n, \quad X_1, \ldots, X_n \text{ are independent,}$$

then

$$\phi_Z(t) = \phi_1(t)\phi_2(t)\cdots\phi_n(t).$$

Example 1.4.2 *Let X_1 ∼Poisson(λ_1) and X_2 ∼Poisson(λ_2),*

$$P(X_1 = r) = \frac{\lambda_1^r}{r!}e^{-\lambda_1}, \quad P(X_2 = r) = \frac{\lambda_2^r}{r!}e^{-\lambda_2}, \quad (r = 0, 1, 2, \ldots).$$

Consider the random variable $Z = X_1 - X_2$,

$$\begin{aligned} X_1 &: \phi_1(t) = \exp[\lambda_1(e^{it} - 1)], \\ X_2 &: \phi_2(t) = \exp[\lambda_2(e^{it} - 1)]. \end{aligned}$$

Using scalar property, the Ch.f. of $-X_2$ is

$$\phi_2(-t) = \exp[\lambda_2(e^{-it} - 1)].$$

Then

$$\phi(t) = \phi_1(t)\phi_2(-t) = \exp[\lambda_1 e^{it} + \lambda_2 e^{-it} - \lambda_1 - \lambda_2].$$

Expanding into power series,

$$\begin{aligned} \phi(t) &= \exp[\lambda_1(1 + it + \frac{(it)^2}{2} + \ldots) + \lambda_2(1 - it + \frac{(it)^2}{2} - \ldots) - \lambda_1 - \lambda_2]. \\ \psi(t) &= \log\phi(t) = (\lambda_1 - \lambda_2)\frac{(it)}{1!} + (\lambda_1 + \lambda_2)\frac{(it)^2}{2!} + (\lambda_1 - \lambda_2)\frac{(it)^3}{3!} + \cdots \end{aligned}$$

we have that all semi-invariants of odd order of Z equal $\lambda_1 - \lambda_2$ and all semi-invariants of even order of Z equal $\lambda_1 + \lambda_2$. In particular, the mean and variance of Z are

$$m_1 = \kappa_1 = \lambda_1 - \lambda_2, \quad \sigma^2 = \kappa_2 = \lambda_1 + \lambda_2.$$

Remark 1.4.3 *The converse of Theorem 1.4.1 is not true. That is, the Ch.f. of the sum of dependent random variables may equal the product the of their Ch.f.s.*

Example 1.4.4 *The joint distribution of the random variable (X, Y) is given by the density:*

$$f(x, y) = \begin{cases} \frac{1}{4}[1 + xy(x^2 - y^2)], & \text{for } |x| \leq 1, |y| \leq 1 \\ 0, & \text{else.} \end{cases}$$

(1) X, Y are dependent.

The marginal distributions in $|x| \leq 1$ and $|y| \leq 1$ are

$$\begin{aligned} f_X(x) &= \int_{-1}^{+1} f(x, y)dy = \frac{1}{4}(y + \frac{1}{2}x^3y^2 - \frac{1}{4}xy^4)|_{y=-1}^{+1} = \frac{1}{2}, \\ f_Y(y) &= \int_{-1}^{+1} f(x, y)dx = \frac{1}{4}(x + \frac{1}{4}x^4y - \frac{1}{2}x^2y^3)|_{x=-1}^{+1} = \frac{1}{2}. \end{aligned}$$

Since

$$f(x, y) \neq f_X(x)f_Y(y) = \frac{1}{4},$$

thus X and Y are NOT independent.

(2) Ch.f. of X and Y.

$$\phi_X(t) = \frac{1}{2}\int_{-1}^{+1} e^{itx}dx = \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_{x=-1}^{+1} = \frac{e^{it}-e^{-it}}{2it} = \frac{\sin t}{t}.$$

Similarly,

$$\phi_Y(t) = \frac{\sin t}{t}.$$

(3) The density of $Z = X + Y$. We compute by the following formula,

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx,$$

where the variables

$$\begin{aligned} -1 &\le x \le 1, \\ -1 &\le y = z - x \le 1 \Leftrightarrow z - 1 \le x \le z + 1. \end{aligned}$$

Then

$$\begin{aligned} \textit{if } z-1 &\le -1, z \le 0 \Rightarrow -1 \le x \le z+1. \\ \textit{if } z-1 &> -1, z > 0 \Rightarrow z-1 \le x \le 1. \end{aligned}$$

The density is given by if $-2 \le z \le 0$,

$$\begin{aligned} f_Z(z) &= \int_{-1}^{z+1} f(x, z-x)dx = \int_{-1}^{z+1} \frac{1}{4}[1 + x\,(z-x)\,(x^2 - (z-x)^2)]dx \\ &= \int_{-1}^{z+1} \frac{1}{4}[1 + \left(xz - x^2\right)(2xz - z^2)]dx = \frac{1}{4}\int_{-1}^{z+1} [1 + 2x^2z^2 - xz^3 - 2x^3z + x^2z^2]dx \\ &= \frac{1}{4}\left(x + x^3z^2 - \frac{1}{2}x^2z^3 - \frac{1}{2}x^4z\right)|_{x=-1}^{z+1} \\ &= \frac{1}{4}\left((z+1) + (z+1)^3z^2 - \frac{1}{2}(z+1)^2z^3 - \frac{1}{2}(z+1)^4z - [-1 - z^2 - \frac{1}{2}z^3 - \frac{1}{2}z]\right) \\ &= \frac{1}{4}(z + 1 + \underleftarrow{z^5} + \underbrace{3z^4} + \underrightarrow{3z^3} + \underline{z^2} - \underleftarrow{\frac{1}{2}z^5}\, \underbrace{-z^4}\, \underrightarrow{-\frac{1}{2}z^3} \\ &\quad \underleftarrow{-\frac{1}{2}z^5}\, \underbrace{-2z^4}\, \underrightarrow{-3z^3}\, \underline{-2z^2} - \frac{1}{2}z + 1 + \underline{z^2} + \underrightarrow{\frac{1}{2}z^3} + \frac{1}{2}z) \\ &= \frac{1}{4}(2 + z). \end{aligned}$$

If $0 \le z \le 2$, *then*

$$\begin{aligned}
f_Z(z) &= \int_{z-1}^{1} f(x, z-x)dx = \int_{z-1}^{1} \frac{1}{4}[1 + 3x^2z^2 - 2x^3z - xz^3]dx \\
&= \frac{1}{4}\left(x + x^3z^2 - \frac{1}{2}x^4z - \frac{1}{2}x^2z^3\right)|_{x=z-1}^{1} \\
&= \frac{1}{4}\left(1 + z^2 - \frac{1}{2}z - \frac{1}{2}z^3 - \left[(z-1) + (z-1)^3z^2 - \frac{1}{2}(z-1)^4z - \frac{1}{2}(z-1)^2z^3\right]\right) \\
&= \frac{1}{4}(1 + \overleftarrow{z^2} \underleftrightarrow{-\frac{1}{2}z} \underrightarrow{-\frac{1}{2}z^3} - z + 1 \underbrace{-z^5} + \underline{3z^4} \underrightarrow{-3z^3} + \overleftarrow{z^2} \\
&\quad + \underbrace{\frac{1}{2}z^5} \underline{-2z^4} + \underrightarrow{3z^3} \underleftarrow{-2z^2} + \underleftrightarrow{\frac{1}{2}z} + \underbrace{\frac{1}{2}z^5} \underline{-z^4} + \underrightarrow{\frac{1}{2}z^3}) \\
&= \frac{1}{4}(2-z).
\end{aligned}$$

Thus, the density is computed by

$$f_Z(z) = \begin{cases} \frac{1}{4}(2+z), & -2 \le z \le 0, \\ \frac{1}{4}(2-z), & 0 < z \le 2, \\ 0, & |z| > 2. \end{cases}$$

The character function can be computed by

$$\begin{aligned}
\phi_Z(t) &= \frac{1}{4}\int_{-2}^{0}(2+z)e^{itz}dz + \frac{1}{4}\int_{0}^{2}(2-z)e^{itz}dz \\
&= \frac{1}{4}\left(\frac{2e^{itz}}{it}|_{z=-2}^{0} + \frac{ze^{itz}}{it}|_{z=-2}^{0} - \frac{1}{it}\int_{-2}^{0} e^{itz}dz\right) \\
&\quad + \frac{1}{4}\left(\frac{2e^{itz}}{it}|_{z=0}^{2} - \frac{ze^{itz}}{it}|_{z=0}^{2} + \frac{1}{it}\int_{0}^{2} e^{itz}dz\right) \\
&= \frac{1}{4}\left(\frac{2}{it} - \frac{2e^{-2it}}{it} - \frac{(-2)e^{-2it}}{it} + \frac{1}{t^2}e^{itz}|_{z=-2}^{0}\right) \\
&\quad + \frac{1}{4}\left(\frac{2e^{2it}}{it} - \frac{2}{it} - \frac{2e^{2it}}{it} - \frac{1}{t^2}e^{itz}|_{z=0}^{2}\right) \\
&= \frac{1}{4}\left(\frac{1}{t^2}(1 - e^{-2it}) - \frac{1}{t^2}(e^{2it} - 1)\right) \\
&= \frac{1}{4t^2}(2 - e^{-2it} - e^{2it}) = \frac{1}{4t^2}(2 - 2\cos 2t) \\
&= \frac{\sin^2 t}{t^2} = \left(\frac{\sin t}{t}\right)^2.
\end{aligned}$$

Thus,

$$\phi_Z(t) = \phi_X(t)\phi_Y(t),$$

however, X *and* Y *are dependent.*

1.5 Determination of the distribution function by the character function

1.5.1 Lévy Theorem

We have known that

$$\begin{aligned} \text{distribution function} &\Rightarrow \text{character function} \\ &\Leftarrow \text{?Yes.} \end{aligned}$$

Lévy Theorem states that the converse is also true. From the Ch.f., we can uniquely determine the distribution function.

Theorem 1.5.1 *The single-valued function $F(x)$ is a distribution function if and only if it is*
(1) nondecreasing,
(2) continuous at least from the left,
(3) satisfies

$$F(-\infty) = 0, F(+\infty) = 1.$$

From this result it follows immediately that the values of the distribution function at the points of continuity determine this function everywhere. The **CDF is almost everywhere continuous**.

Proposition 1.5.2 *The set of points of discontinuity is at most countable.*

Proof. Denote by H_n set of points at which the distribution function $F(x)$ has a jump not smaller than $1/n$. Then we have

$$H = H_1 \cup H_2 \cup \cdots$$

For every n the set H_n is finite; hence the set H is at most countable. ∎

Theorem 1.5.3 *(Lévy Theorem) Let $F(x)$ and $\phi(t)$ be CDF and Ch.f. of the random variable X. If $a+h$ and $a-h$ $(h>0)$ are continuity points of the CDF $F(x)$, then*

$$F(a+h) - F(a-h) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(ht)}{t} e^{-ita} \phi(t) dt. \tag{1.5}$$

Before proving the theorem, we first see the application of the theorem. Since a and h are arbitrary, equation (1.5) gives

$$F(x_2) - F(x_1) = P(x_1 \le X \le x_2),$$

for arbitrary continuity points x_1 and x_2. Let $x = x_2$ be a given continuity point and let $x_1 \to -\infty$. Hence the sequence of differences

$$F(x) - F(x_1) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(\frac{x-x_1}{2} t)}{t} e^{-it\frac{x+x_1}{2}} \phi(t) dt,$$

is determined by the Ch.f. and is convergent to $F(x)$. Thus, the CDF $F(x)$ is determined at every continuity point. Hence by Proposition 1.5.2, it is determined almost everywhere.

Proof. Only consider X of continuous type with density function $f(x)$. Denote

$$J = \frac{1}{\pi}\int_{-T}^{T} \frac{\sin(ht)}{t} e^{-ita}\phi(t)dt.$$

By definition of the Ch.f.,

$$\begin{aligned} J &= \frac{1}{\pi}\int_{-T}^{T} \frac{\sin(ht)}{t} e^{-ita} \int_{-\infty}^{\infty} e^{itx} f(x)dxdt \\ &= \frac{1}{\pi}\int_{-T}^{T} \left[\int_{-\infty}^{\infty} \frac{\sin(ht)}{t} e^{it(x-a)} f(x)dx\right] dt. \end{aligned}$$

By Fubini's Theorem, since

$$\begin{aligned} \int_{-\infty}^{\infty} \left|\frac{\sin(ht)}{t} e^{it(x-a)} f(x)\right| dx &= \int_{-\infty}^{\infty} \left|\frac{\sin(ht)}{t}\right| f(x)dx \\ &\leq h\int_{-\infty}^{\infty} f(x)dx = h. \end{aligned}$$

$$\frac{1}{\pi}\int_{-T}^{T} \left[\int_{-\infty}^{\infty} \left|\frac{\sin(ht)}{t} e^{it(x-a)} f(x)\right| dx\right] dt \leq \frac{h}{\pi}2T < \infty.$$

Then we can interchange the order of integration,

$$\begin{aligned} J &= \frac{1}{\pi}\int_{-\infty}^{\infty} \left[\int_{-T}^{T} \frac{\sin(ht)}{t} e^{it(x-a)} f(x)dt\right] dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \left[\int_{-T}^{T} \underbrace{\frac{\sin(ht)}{t}}_{\text{even}} \left\{\underbrace{\cos[(x-a)t]}_{\text{even}} + i\underbrace{\sin[(x-a)t]}_{\text{odd}}\right\} f(x)dt\right] dx \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} \left[\int_{0}^{T} \frac{\sin(ht)}{t} \{\cos[(x-a)t]\} f(x)dt\right] dx. \end{aligned}$$

By triangular formula, $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ where $A = ht$, $B = xt - at$, we have

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{\pi}\left[\int_{0}^{T} \left\{\frac{\sin((x-a+h)t)}{t} - \frac{\sin((x-a-h)t)}{t}\right\} dt\right]}_{:=g(x,T)} f(x)dx \\ &= \int_{-\infty}^{\infty} g(x,T)f(x)dx. \end{aligned}$$

Since (1) the integral $\int_0^T \frac{\sin x}{x}dx$ is bounded for all T,
(2) $\lim_{T\to\infty}\int_0^T \frac{\sin x}{x}dx = \frac{\pi}{2}$,
then (1) $|g(x,T)|$ is bounded for all T, assuming that $|g(x,T)| \leq C_g$,
(2) $\int_{-\infty}^{\infty} |g(x,T)f(x)|\, dx \leq C_g \int_{-\infty}^{\infty} f(x)dx = C_g$,
which mean that $J = \int_{-\infty}^{\infty} g(x,T)f(x)dx$ is uniformly convergent with respect to T. Hence we can interchange $\lim_{T\to\infty}$ and $\int_{-\infty}^{\infty}$, to obtain

$$\lim_{T\to\infty} J = \lim_{T\to\infty}\int_{-\infty}^{\infty} g(x,T)f(x)dx = \int_{-\infty}^{\infty} \lim_{T\to\infty} g(x,T)f(x)dx.$$

Finally we need to compute $\lim_{T\to\infty} g(x,T)$,

$$\lim_{T\to\infty} g(x,T) = \lim_{T\to\infty} \frac{1}{\pi}\int_0^T \frac{\sin((x-a+h)t)}{t}dt - \lim_{T\to\infty} \frac{1}{\pi}\int_0^T \frac{\sin((x-a-h)t)}{t}dt.$$

Notice that

$$\lim_{T\to\infty} \frac{1}{\pi}\int_0^T \frac{\sin(\alpha t)}{t}dt = \lim_{T\to\infty} \frac{1}{\pi}\int_0^T \frac{\sin(\alpha t)}{\alpha t}d\alpha t = \begin{cases} \frac{1}{2}, & \text{if } \alpha > 0, \\ 0, & \text{if } \alpha = 0, \\ -\frac{1}{2}, & \text{if } \alpha < 0, \end{cases}$$

where the convergence is uniform with respect to α if $|\alpha| = |x-a\pm h| > \delta > 0$. (Here is the proof for the uniform convergence using Dirichlet discriminant. (1) $\left|\int_0^T \sin(\alpha t)dt\right| = \left|\frac{1}{\alpha}(-\cos\alpha T + 1)\right| \le \frac{2}{|\alpha|} < \frac{2}{\delta}$ for $|\alpha| > \delta$. (2) $\frac{1}{t} \to 0$ uniformly w.r.t. α. Thus, $\lim_{T\to\infty} \frac{1}{\pi}\int_0^T \frac{\sin(\alpha t)}{t}dt$ is uniformly convergent w.r.t. α for $|\alpha| > \delta > 0$.) Hence,

$$\lim_{T\to\infty} g(x,T) = \begin{cases} (-\frac{1}{2}) - (-\frac{1}{2}) = 0, & \text{if } x > a-h, \\ 0 - (-\frac{1}{2}) = \frac{1}{2}, & \text{if } x = a-h, \\ \frac{1}{2} - (-\frac{1}{2}) = 1, & \text{if } x-h < x < a+h, \\ \frac{1}{2} - 0 = \frac{1}{2}, & \text{if } x = a+h, \\ \frac{1}{2} - \frac{1}{2} = 0, & \text{if } x > a+h. \end{cases}$$

Then

$$\begin{aligned} \lim_{T\to\infty} J &= \int_{-\infty}^{\infty} \lim_{T\to\infty} g(x,T)f(x)dx \\ &= \int_{a-h}^{a+h} f(x)dx = F(a+h) - F(a-h). \end{aligned}$$

■

Remark 1.5.4 *For **discrete type random variables**, the proof is similar. It's only necessary to replace the integrals by series.*

1.5.2 Simplification in Specific Cases

Proposition 1.5.5 *Moreover, if the Ch.f. $\phi(t)$ is **absolutely integrable** over $(-\infty,\infty)$, i.e., $\int_{-\infty}^{\infty}|\phi(t)|dt < \infty$, then the corresponding density function $f(x)$ can be determined by $\phi(t)$.*

Proof. Since $\phi(t)$ is absolutely integrable, the improper integral (1.5) exists,

$$\lim_{T\to\infty} \frac{1}{\pi}\int_{-T}^T \left|\frac{\sin(ht)}{t}e^{-ita}\phi(t)\right| dt \le \frac{h}{\pi}\int_{-\infty}^{\infty} |\phi(t)|\, dt < \infty. \tag{1.6}$$

In (1.5) dividing both sides by $2h$,

$$\frac{F(x+h) - F(x-h)}{2h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(ht)}{ht}e^{-itx}\phi(t)dt.$$

Since RHS is uniformly convergent w.r.t. h (due to Weierstrass comparison discriminant and (1.6)), and also $\frac{\sin(ht)}{ht}e^{-itx}\phi(t)$ is continuous w.r.t. t and h (due to the continuity of the Ch.f. $\phi(t)$), then we can interchange

$\lim_{h\to 0}$ and $\int_{-\infty}^{\infty}$,

$$\begin{aligned}\lim_{h\to 0}\frac{F(x+h)-F(x-h)}{2h} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{h\to 0}\frac{\sin(ht)}{ht}e^{-itx}\phi(t)dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi(t)dt,\end{aligned}$$

where the RHS is well-defined (so that LHS is well-defined and F is differentiable). Thus,

$$F'(x) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi(t)dt, \tag{1.7}$$

where the pdf $f(x)$ is the Fourier inverse transform of the Ch.f. $\phi(t)$. From the absolute and uniform convergence of the RHS, it follows that $F'(x)$ exists. Moreover, since $e^{-itx}\phi(t)$ is continuous w.r.t. x and t, and $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi(t)dt$ is uniformly convergent w.r.t. x, then $f(x)$ is a **continuous function**. ■

Therefore, equation (1.5) allows us to determine the density $f(x)$ from the Ch.f. $\phi(t)$ under the assumption that $\phi(t)$ is absolutely integrable.

Example 1.5.6 *(continuous random variable) The Ch.f. of random variable X is given by*

$$\phi(t) = e^{-t^2/2}.$$

Find the density function of X.

Solution. Notice that $\phi(t)$ is absolutely integrable, so that

$$\begin{aligned}f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}e^{-\frac{t^2}{2}}dt\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{(t+ix)^2}{2}}e^{\frac{(ix)^2}{2}}dt\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{(t+ix)^2}{2}}dt = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.\end{aligned}$$

This is the density of the standard normal distribution.

Example 1.5.7 *(discrete random variable) If X is of discrete type and can take on only integer values, then its probability mass function (pmf) can be easily obtained from the Ch.f.*

For every integer k, let

$$p_k = P(X=k),$$

where $p_k \geq 0$ for all k. Then we have

$$\phi(t) = \sum_{k'=-\infty}^{\infty}p_{k'}e^{ik't},$$

*which is the **Fourier series**. Multiplying by e^{-ikt},*

$$e^{-ikt}\phi(t) = \sum_{\substack{k'=-\infty\\ k'\neq k}}^{\infty}p_{k'}e^{-i(k-k')t} + p_k.$$

Integrating $\int_{-\pi}^{\pi}dt$, using the fact for $k'\neq k$, we have

$$\int_{-\pi}^{\pi}e^{-it(k-k')}dt = 0.$$

Thus,

$$p_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\phi(t)dt,$$

which can be regarded as the ***Fourier coefficient****.*

Example 1.5.8 *Moreover, if Ch.f. $\phi(t)$ is not absolutely integrable but is a periodic function, then the corresponding random variable is discrete. If the period of $\phi(t)$ is 2π, then the random variable takes on only integer values (spacing with distance 1). In general, if the period of $\phi(t)$ is ω, then the random variable has a lattice distribution spacing with distance $2\pi/\omega$.*

1.5.3 Ch.f. in a Finite Inverval

Gnedenko proved that the values of the Ch.f. in a finite interval do not uniquely determine the distribution function. (This means that the distribution function cannto be uniquely determined by the values of the Ch.f. in any finite interval.)

(1) Let us find the density function of X, whose Ch.f. is

$$\phi_1(t) = \begin{cases} 1-|t|, & \text{for } |t| \le 1, \\ 0, & \text{for } |t| > 1. \end{cases}$$

Notice that $\phi_1(t)$ is absolutely integrable over $(-\infty, \infty)$. Thus,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi_1(t)dt = \frac{1}{2\pi}\int_{-1}^{0}(1+t)e^{-itx}dt + \frac{1}{2\pi}\int_{0}^{1}(1-t)e^{-itx}dt.$$

First, one can compute $\int e^{-itx}dt = \frac{e^{-itx}}{-ix}$ and
$\int te^{-itx}dt = \int \frac{tde^{-itx}}{-ix} = -\frac{1}{ix}\left(te^{-itx} - \int e^{-itx}dt\right) = -\frac{1}{ix}\left(te^{-itx} + \frac{e^{-itx}}{ix}\right)$. Then

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\left(\frac{e^{-itx}}{-ix}|_{t=-1}^{0} - \frac{te^{-itx}}{ix}|_{t=-1}^{0} + \frac{1}{x^2}e^{-itx}|_{t=-1}^{0}\right) \\ &\quad + \frac{1}{2\pi}\left(\frac{e^{-itx}}{-ix}|_{t=0}^{1} + \frac{te^{-itx}}{ix}|_{t=0}^{1} - \frac{1}{x^2}e^{-itx}|_{t=0}^{1}\right) \\ &= \frac{1}{2\pi}\left(\frac{e^{-ix}}{-ix} - \frac{e^{ix}}{-ix} - \frac{e^{ix}}{ix} + \frac{1}{x^2} - \frac{1}{x^2}e^{ix} + \frac{e^{-ix}}{ix} - \frac{1}{x^2}\left(e^{-ix}-1\right)\right) \\ &= \frac{1}{2\pi x^2}\left(1 - e^{ix} + 1 - e^{-ix}\right) = \frac{1-\cos x}{\pi x^2}. \end{aligned}$$

(2) Let us consider Y of disrete type with pmf by

$$\begin{aligned} P(Y &= 0) = \frac{1}{2}, \\ P(Y &= (2k-1)\pi) = \frac{2}{(2k-1)^2\pi^2}, \quad (k = 0, \pm 1, \pm 2, \cdots) \end{aligned}$$

The Ch.f. is

$$\begin{aligned} \phi_1(t) &= \frac{1}{2} + \sum_{k=-\infty}^{\infty}\frac{2}{(2k-1)^2\pi^2}e^{it(2k-1)\pi} \\ &= \frac{1}{2} + \frac{2}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{\cos[(2k-1)\pi t] + i\sin[(2k-1)\pi t]}{(2k-1)^2} \\ &= \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)\pi t]}{(2k-1)^2}. \end{aligned}$$

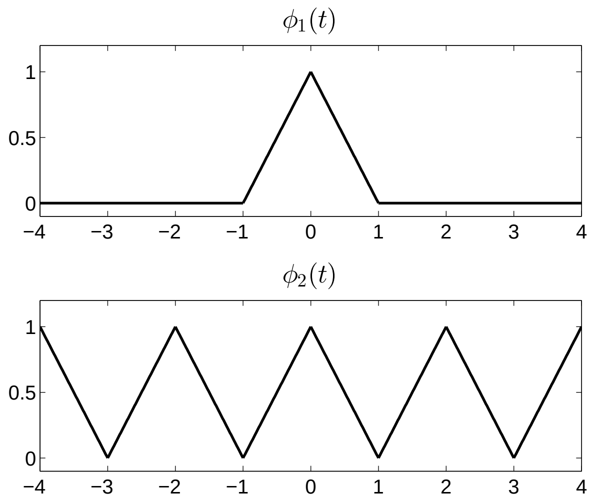

Figure 1.1: Comparison of Ch.f.s $\phi_1(t)$ and $\phi_2(t)$. Notice that $\phi_1(t) = \phi_2(t)$ for $t \in (-1, 1)$. However, $\phi_1(t) \neq \phi_2(t)$ for all $t \in (-\infty, \infty)$.

We shall show that for $|t| \leq 1$, we have

$$\phi_1(t) = \phi_2(t).$$

Expanding $\psi(t) = |t|$ in the interval for $|t| \leq 1$ in a Fourier series, we have

$$\psi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t,$$

where we notice that $b_n = 0$ for all n since $\psi(t)$ is even. The other coefficients are

$$\begin{aligned} \frac{a_0}{2} &= \int_0^1 t dt = \frac{1}{2}, \\ a_n &= 2\int_0^1 t \cos n\pi t dt = \left[\frac{2t \sin n\pi t}{n\pi}\right] |_{t=0}^1 - \frac{2}{n\pi} \int_0^1 \sin n\pi t dt \\ &= -\frac{2}{n\pi} \left[\frac{-\cos n\pi t}{n\pi}\right] |_{t=0}^1 = \frac{2\left(\cos n\pi - 1\right)}{n^2\pi^2}. \end{aligned}$$

Then we see that

$$\begin{array}{ll} \text{For even } n, & a_n = 0. \\ \text{For odd } n = 2k-1, & a_{2k-1} = -\frac{4}{(2k-1)^2\pi^2}. \end{array}$$

Thus,

$$\psi(t) = |t| = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos\left[(2k-1)\pi t\right]}{(2k-1)^2}.$$

For $|t| \leq 1$, we find that

$$\underbrace{\phi_1(t)}_{\text{continuous}} = \underbrace{\phi_2(t)}_{\text{discrete}},$$

where $\phi_2(t)$ is not absolutely integrable and is a periodic function whereas $\phi_1(t)$ is absolutely integrable and $\phi_1(t) = 0$ for $|t| > 1$.

1.6 Character Function of Multi-Dimensional Random Variables

Let (X, Y) be a two-dimensional random variables and let $F(X, Y)$ be its distribution function. Let t, s be two real numbers. The Ch.f. is

$$\phi(t, s) = Ee^{i(tX+sY)}.$$

Example 1.6.1 *There are 4 points* $(+1, +1)(+1, -1)(-1, +1)(-1, -1)$. *The probability is*

$$\begin{aligned} P(X &= +1, Y = +1) = \frac{1}{3}, \quad P(X = +1, Y = -1) = \frac{1}{3}, \\ P(X &= -1, Y = +1) = \frac{1}{6}, \quad P(X = -1, Y = -1) = \frac{1}{6}. \end{aligned}$$

The Ch.f. is

$$\begin{aligned} \phi(t, s) &= E(e^{i(tX+sY)}) = \frac{1}{3}e^{i(t+s)} + \frac{1}{3}e^{i(t-s)} + \frac{1}{6}e^{i(-t+s)} + \frac{1}{6}e^{i(-t-s)} \\ &= \frac{1}{3}e^{it}\left(e^{is} + e^{-is}\right) + \frac{1}{6}e^{-it}\left(e^{is} + e^{-is}\right) \\ &= \frac{1}{3}e^{it}2\cos s + \frac{1}{6}e^{-it}2\cos s \\ &= \frac{2}{6}\cos s(2e^{it} + e^{-it}) = \frac{\cos s}{3}(3\cos t + i\sin t). \end{aligned}$$

Proposition 1.6.2 *(1)* $\phi(0, 0) = Ee^{i(0X+0Y)} = 1$.
(2) $|\phi(t, s)| = \left|Ee^{i(tX+sY)}\right| \le E\left|e^{i(tX+sY)}\right| = 1$.
(3) $\phi(-t, -s) = Ee^{-i(tX+sY)} = \overline{\phi(t, s)}$.
(4) If all the moments of order k of a multi-dimensional random variable exist, then the derivatives

$$\frac{\partial^k \phi(t, s)}{\partial t^{k-l}\partial s^l} \quad \text{for } l = 0, 1, \ldots, k,$$

exist and can be obtained from the formula

$$\begin{aligned} \frac{\partial^k \phi(t, s)}{\partial t^{k-l}\partial s^l} &= \frac{\partial^k}{\partial t^{k-l}\partial s^l}Ee^{i(tX+sY)} \\ &= i^k E\left[X^{k-l}Y^l e^{i(tX+sY)}\right]. \end{aligned}$$

The moment $m_{k-l,l}$ is

$$m_{k-l,l} = E\left[X^{k-l}Y^l\right] = \frac{1}{i^k}\left[\frac{\partial^k \phi(t, s)}{\partial t^{k-l}\partial s^l}\right]|_{t=0,s=0}.$$

(5) By putting $t = 0$, we have the Ch.f. of Y,

$$\phi(0, s) = E(e^{isY}) = \phi_Y(s).$$

Similarly,

$$\phi(t, 0) = E(e^{itX}) = \phi_X(t).$$

Example 1.6.3 *The moment can be computed by*

$$\begin{aligned} m_{10} &= \frac{1}{i}\left[\frac{\partial\phi(t, s)}{\partial t}\right]|_{t=0,s=0}, \quad m_{01} = \frac{1}{i}\left[\frac{\partial\phi(t, s)}{\partial s}\right]|_{t=0,s=0}, \\ m_{20} &= \frac{1}{i^2}\left[\frac{\partial^2\phi(t, s)}{\partial t^2}\right]|_{t=0,s=0}, \quad m_{11} = \frac{1}{i^2}\left[\frac{\partial\phi(t, s)}{\partial t\partial s}\right]|_{t=0,s=0}, \\ m_{02} &= \frac{1}{i^2}\left[\frac{\partial^2\phi(t, s)}{\partial s^2}\right]|_{t=0,s=0}. \end{aligned}$$

Theorem 1.6.4 *Let $\phi(t,s)$ be the Ch.f. of the random variable (X,Y). If the rectangle $(a-h \le X \le a+h$, $b-g \le Y \le b+g)$ is a continuity rectangle (see Definition 2.5.6 in Fiesz), then*

$$
\begin{aligned}
P(a-h &\le X \le a+h, b-g \le Y \le b+g) \\
&= \lim_{T\to\infty} \frac{1}{\pi^2} \int_{-T}^{+T} \int_{-T}^{+T} \frac{\sin(ht)}{t} \frac{\sin(gs)}{s} e^{-i(at+bs)} \phi(t,s) dt ds. \qquad (1.8)
\end{aligned}
$$

Theorem 1.6.5 *Let $F(x,y), F_X(x), F_Y(y)$ be distribution functions of $(X,Y), X, Y$, respectively. Let $\phi(t,s)$, $\phi_X(t)$, $\phi_Y(s)$ be Ch.f.s of $(X,Y), X, Y$, respectively. The random variables X and Y are then independent if and only if*

$$\phi(t,s) = \phi_X(t)\phi_Y(s),$$

holds for all t and s.

Proof. $\Rightarrow$ X and Y are independent, then

$$\phi(t,s) = Ee^{i(tX+sY)} = Ee^{itX}Ee^{isY} = \phi_X(t)\phi_Y(s).$$

$\Leftarrow$ Now $\phi(t,s) = \phi_X(t)\phi_Y(s)$, then by definition,

$$
\begin{aligned}
P(a-h &\le X \le a+h, b-g \le Y \le b+g) \\
&= \lim_{T\to\infty} \frac{1}{\pi^2} \int_{-T}^{+T} \int_{-T}^{+T} \frac{\sin(ht)}{t} \frac{\sin(gs)}{s} e^{-i(at+bs)} \phi(t,s) dt ds \\
&= \lim_{T\to\infty} \left(\frac{1}{\pi} \int_{-T}^{+T} \frac{\sin(ht)}{t} e^{-iat} \phi_X(t) dt \right) \lim_{T\to\infty} \left(\frac{1}{\pi} \int_{-T}^{+T} \frac{\sin(gs)}{s} e^{-ibs} \phi_Y(s) ds \right) \\
&= P(a-h \le X \le a+h) P(b-g \le Y \le b+g),
\end{aligned}
$$

which is valid for arbitrary continuity rectangles. ■

The following Cramér-Wold theorem is useful in the theory of random vectors.

Theorem 1.6.6 *The CDF $F(x,y)$ of a two-dimensional random variable (X,Y) is uniquely determined by the class of all one-dimensional distribution functions of $tX+sY$, where t,s run over all possible real values.*

Proof. Suppose we are given for all real t and s, the Ch.f. $\phi_Z(v)$ of $Z = tX+sY$,

$$\phi_Z(v) = Ee^{iv(tX+sY)}.$$

Let $v=1$, we obtain $Ee^{i(tX+sY)}$ which is the Ch.f. $\phi(t,s)$ of the distribution function $F(x,y)$. Since $\phi(t,s)$ uniquely determines $F(x,y)$, then the proof is complete. ■

（概率母函数）

1.7 Probability-Generating Functions

When investigating random variables which take on only the integers $k = 0,1,2,\cdots$, it is simpler to deal with probability generating functions than with Ch.f.s. Let X be a random variable and let

$$p_k = P(X=k) \quad (k = 0,1,2,\cdots),$$

where $\sum_k p_k = 1$.

Definition 1.7.1 *The function defined by the formula*

$$\psi(s) = \sum_k p_k s^k = Es^X, \tag{1.9}$$

where $-1 \leq s \leq 1$ *is called the probability generating funciton of* X.

Remark 1.7.2 *Since* $\psi(1) = \sum_k p_k = 1 < \infty$, $\psi(s)$ *is absolutely and uniformly convergent in* $|s| \leq 1$. *Thus* $\psi(s)$ *is continuous.*

Remark 1.7.3 *It determines the pmf uniquely since* $\psi(s)$ *can be represented in a unique way as a power series of the form (1.9).*

Example 1.7.4 X *has a binomial distribution, that is,*

$$p_k = C_n^k p^k (1-p)^{n-k} \quad (k = 0, 1, \ldots, n)$$

Thus,

$$\psi(s) = \sum_{k=0}^{n} C_n^k p^k (1-p)^{n-k} s^k = (ps + q)^n.$$

Example 1.7.5 X *has a Poisson distribution, that is,*

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!} \quad (k = 0, 1, 2, \ldots).$$

Thus,

$$\psi(s) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k = e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}.$$

Proposition 1.7.6 *The moments of* X *can be determined by the derivatives at the point* 1 *of the generating function.*

Example 1.7.7 *Let* $\psi(s)$ *be the probability generating function. Then*

$$\begin{aligned} \psi'(s) &= \sum_k k p_k s^{k-1}, \\ \psi''(s) &= \sum_k k(k-1) p_k s^{k-2}. \end{aligned}$$

Thus,

$$\begin{aligned} \psi'(1) &= \sum_k k p_k = EX, \\ \psi''(1) &= \sum_k k(k-1) p_k = EX^2 - EX. \end{aligned}$$

Then

$$EX^2 = \psi''(1) + \psi'(1).$$

Bibliography

[1] M. Fisz. Probability theory and mathematical statistics. *(No Title)*, 1963.